(7) "Curve fitting" by Professor J. R. MUSSELMAN of Johns Hopkins University.

(8) "On Kichewsky's method of fitting frequency curves" by E. W. WOOLARD of the U. S. Weather Bureau.

2. Professor Murnaghan gave an outline of a presentation of the theory of determinants using the generalized Kronecker symbol $\delta^{r_1\cdots r_m}_{s_1\cdots s_m}$. This symbol is defined by the following properties. The letters $r_1\cdots r_m$, $s_1\cdots s_m$, each being capable of taking any one of the n values $1, 2, \cdots, n$ $(m \leqq n)$, $\delta^{r_1\cdots r_m}_{s_1\cdots s_m}$ is alternating both in the superscripts and subscripts; *i.e.*, an interchange of any two superscripts or of any two subscripts changes the sign but not the numerical value of the symbol. If the values assigned to $(r_1\cdots r_m)$ are all distinct, as are also the values assigned to $(s_1\cdots s_m)$, the symbol has the value zero if the group $(r_1\cdots r_m)$ is different from the group $(s_1\cdots s_m)$ and if these groups are the same, it has the value ± 1 according as the permutations $(r_1\cdots r_m)$, $(s_1\cdots s_m)$ of the same m numbers are of the same class or not. Then the determinant $|a_{rs}|$ may be defined as $= \delta^{\alpha_1\cdots\alpha_n}_{s_1\cdots s_n} a_{\alpha_1 s_1}\cdots a_{\alpha_n s_n}$ where the $\alpha_1\cdots\alpha_n$ are summation or umbral symbols. The theorems on the differentiation of determinants, Laplace's expansion, product of two determinants, etc., follow immediately from this definition. Conversely, the symbol

$$\delta^{r_1\cdots r_m}_{s_1\cdots s_m} = \begin{vmatrix} \delta^{r_1}_{s_1} & \cdots & \delta^{r_1}_{s_m} \\ \cdot & \cdots & \cdot \\ \delta^{r_m}_{s_1} & \cdots & \delta^{r_m}_{s_m} \end{vmatrix},$$

where δ^r_s is the usual Kronecker symbol. (*To be published in the* MONTHLY.)

3. Vincenzo Viviani, a Florentine mathematician, proposed a problem to his contemporaries in the form of an enigma: To cut out four equal windows from a hemispherical dome so that the area of the surface left should be exactly that of a square (*Acta Eruditorum*, 5 April, 1692). Leibnitz, James Bernoulli, John Wallis, David Gregory reached the solution and Viviani gave a construction without proof. This solution now appears in the elementary calculus text in some such form as this: To show that the cylinders $x^2 + y^2 = \pm Rx$, cut out from the hemisphere $z = +\sqrt{R^2 - x^2 - y^2}$ such an area that the remainder is $4R^2$, *i.e.*, the square on a diameter. Leibnitz, Bernoulli and the rest placed the cylinders with axes in the base of the hemisphere $(x^2 + z^2 = \pm Rx)$, thus cutting windows of somewhat semicircular appearance.

Taking the origin at the center of the sphere and using ordinary cylindrical coördinates, we see that the integral $R\int_0^{2\pi} \sqrt{R^2 - [f(\theta)]^2}\,d\theta$ gives the area of the surface of the hemisphere outside the cylinder $\rho = f(\theta)$ which stands upon the base of the hemisphere. Considering the equation

$$4R^2 = R\int_0^{2\pi} \sqrt{R^2 - [f(\theta)]^2}\,d\theta,$$

we see that $f(\theta) = R\cos 2n\theta$ is a solution, and that the cylinder $\rho = R\cos 2\theta$

cuts four equal windows from the dome and that this solution is more satisfactory than the classic one. It is interesting to note that the cylinders $\rho = R \cos 2n\theta$ (n any integer) cut a constant area (and also constant volume) from the hemisphere $\rho^2 + z^2 = R^2$. Similarly do the cylinders $\rho = R \cos (2n + 1)\theta$. Thus it is possible to cut any even number of windows and leave a remainder $4R^2$. In case $n = 0$ we get the classic solution of the problem.

4. Euclidean geometry can be introduced by a system of axioms involving vectors. The fundamental objects are points $A, B, C, \cdots$. Vectors are introduced as differences between points, $B - A, C - A, \cdots$. Vectors having the same initial point, *i.e.*, the same subtrahend, form a bundle; operations of addition and multiplication for the vectors of the same bundle are characterized by the laws of these operations. A vector $B - A$ is called equal to the vector $D - C$ if $B - A = D - A - (C - A)$. The last two axioms are these: From $A = B$ follows $B = A$ and from $A = B$, $B = C$ follows $A = C$. If we drop the last of these axioms, we obtain a much more general geometry of which the Riemann Geometry is a special case.

5. Professor Bramble presented a discussion of the probabilities of two players making certain scores with alternate throws of a single die. The particular interest of the problem lay in the variety of distinct methods of solution.

6. A brief historical introduction contained an account of the rise and development of the Gothic window and of the various forms of ornamentation used by successive generations of architects and culminating in the beautiful and intricate tracery of the great rose windows of the cathedrals built in the twelfth, thirteenth and fourteenth centuries. The rest of the paper dealt with the mathematical expression representing the general characteristics of that particular type of Gothic ornamentation which is known as "Flowing Tracery." The following parametric equations were found to represent quite faithfully this style of tracery:

$$\theta = \frac{c}{p}(t - \tfrac{1}{2} \sin 2t),$$

$$\rho = a(1 - k \cos t),$$

where θ and ρ are polar coördinates, c and p are relatively prime integers, a is an arbitrary constant unrestricted in value, and k is equal to or less than unity. It was shown that the character and shape of the curve depend on the three constants c, p and k, while the constant a affects only the size of the curve. A number of window designs plotted from the above equations were exhibited, the designs being cut out from paper and backed with a material resembling stained glass.

7. Dr. Musselman discussed the mathematical theory underlying the expression obtained by Charlier for fitting curves to statistical data. In particular, he presented the contributions to this theory by Thiele, Gram, and Charlier.

8. The two systems of skew curves in most general use for empirically fitting observed frequency distributions are the Pearson curves and the Gram-Charlier

expansion; a number of other systems have been proposed from time to time, and in this paper a recent method due to S. Krichewsky of the Egyptian Ministry of Public Works was described. An application to the frequency distribution of rainfall amounts at Washington, D. C., showed that the method gave as good a fit as that of Pearson, and with much less labor. Brief mention was made of the practical uses of frequency curves in meteorology, civil engineering, actuarial work, etc., and some precautions necessary to their intelligent use pointed out. The paper has been published in the *Monthly Weather Review*, 52, 91–94, February 1924.

The sixteenth meeting of the Section was held in Baltimore, December 6, 1924. The program will appear later.

HARRY ENGLISH, *Secretary-Treasurer.*

EIGHTH ANNUAL MEETING OF THE MISSOURI SECTION.

The eighth annual meeting of the Missouri Section of the Mathematical Association of America was held at the Junior College of Kansas City on Saturday morning, November 15, 1924, in affiliation with the annual meeting of the Missouri State Teachers Association.

The attendance was twenty-three including the following thirteen members of the Association: A. C. Andrews, Theodosia T. Callaway, L. H. Cutting, B. F. Finkel, R. R. Fleet, A. H. Huntington, W. A. Luby, A. D. Pierson, P. R. Rider, J. H. Scarborough, E. Stephens, R. A. Wells, Meta Wood.

The session was presided over by Professor R. R. Fleet, chairman of the Section. The following officers were elected for 1925: Chairman, R. A. WELLS, Park College; Vice-chairman, A. C. ANDREWS, Manual Training High School, Kansas City; Secretary-Treasurer, P. R. RIDER, Washington University. A number of the members of the Section attended the luncheon for mathematics teachers at the Coates House on Friday, November 14, and the meeting of the department of mathematics of the State Teachers Association, which was held at the Coates House on Friday afternoon.

The following four papers were read:

1. "Symbolic calculus" by Mr. EUGENE STEPHENS, Washington University.
2. "Service mathematics" by Professor THEODOSIA T. CALLAWAY, Stephens Junior College.
3. "How and what should freshmen be taught?" by Professor R. R. FLEET, William Jewell College.
4. "Simple illustrations of certain types of statistical series" by Professor P. R. RIDER, Washington University.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Mr. Stephens gave a historical introduction to the subject of symbolic

operators and their application to the operations of the calculus and to differential equations, together with a bibliography of papers prior to 1900. (This is an introductory paper only; others will follow, developing the subject completely.)

2. Mrs. Callaway's paper reviewed briefly the efforts made at Stephens College to locate and remove the mathematical difficulties of students in elementary clothing and foods classes. A careful study was made to determine definitely what mathematical concepts and processes are encountered in these courses and the degree of difficulty to which each process is carried. The study shows that no mathematics beyond that acquired in the first seven or eight grades is necessary. Experimentation has shown that tests and drill sheets based on the results of this study easily locate and remove difficulties in mathematical manipulation; but this inability to handle the mathematical processes is not nearly so prevalent as the inability to analyze problems and propose solutions. The paper ended with a plea for teachers of all subjects, and especially teachers of mathematics, to put more emphasis on methods of attacking problems, both general and mathematical.

3. Professor Fleet spoke of the present difficulty of preparing college courses in mathematics suitable to all classes of students enrolling in that subject. Local statistics were cited to show that high school students had not caught the spirit of the *Report of the National Committee* (1919) concerning the essentials necessary for the pursuit of a college course in mathematics. It was suggested that a list of those subjects in elementary mathematics needing especial emphasis be made known to the secondary schools.

4. Professor Rider's paper gave some simple examples of Bernoulli, Poisson, and Lexis series, and illustrated the formulas for mean and standard deviation in the three types.

P. R. RIDER, *Secretary-Treasurer.*

TENTH REGULAR MEETING OF THE KANSAS SECTION.

The tenth regular meeting of the Kansas Section was held at the Central High School, Topeka, Kansas, February 2, 1924. All of the regular officers were absent on account of illness or removal from the state. The meeting was called to order by Professor C. H. Ashton, University of Kansas, and Professor W. H. Garrett, Baker University, was elected temporary Chairman and Professor G. W. Smith, University of Kansas, temporary Secretary.

Among those attending were the following twenty members of the M. A. A.: C. H. Ashton, Florence Black, Wealthy Babcock, R. H. Carpenter, Lucy T. Dougherty, E. F. Farner, W. H. Garrett, W. A. Harshbarger, W. C. Janes, S. Lefschetz, C. F. Lewis, O. B. Loewen, Thirza A. Mossman, H. S. Myers, B. L. Remick, G. W. Smith, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected for the coming year: Chairman, Professor J. J. WHEELER, University of Kansas, Vice-chairman, LUCY T. DOUGHERTY,

Kansas City Junior College, Secretary-Treasurer, Professor U. G. MITCHELL, University of Kansas.

The following papers were presented:

(1) "Some space curves and their intrinsic equations" by Professor R. H. CARPENTER, Iola Junior College.

(2) "Holditch's theorem and Kempe's theorem for areas" by Professor W. T. STRATTON, Kansas State Agricultural College.

(3) "An hour with Einstein—Relativity: what it is and what it is not" by Professor S. LEFSCHETZ, University of Kansas.

A general discussion of Professor Lefschetz's paper was led by Professor E. B. Stouffer, University of Kansas.

Abstracts of papers, numbered as in the above list, follow:

1. This paper discussed certain cylindrical helices defined by their intrinsic equations and showed the nature of the surfaces upon which they lie.

2. Proofs of these two theorems were followed by a discussion of Woolhouse's and Elliott's extensions of the first and the many interesting variations which arise in connection with the second.

3. This paper consisted in a discussion of the growth of the Einstein theory, mainly in its more mathematical aspects, culminating in the assumption of the permanence of a ds^2 and some of the consequences of this assumption.

The eleventh regular meeting was held in Topeka in connection with the meeting of the State Council of Administration, February 7, 1925. Report of this meeting will be given at a later date.

U. G. MITCHELL, *Secretary-Treasurer.*

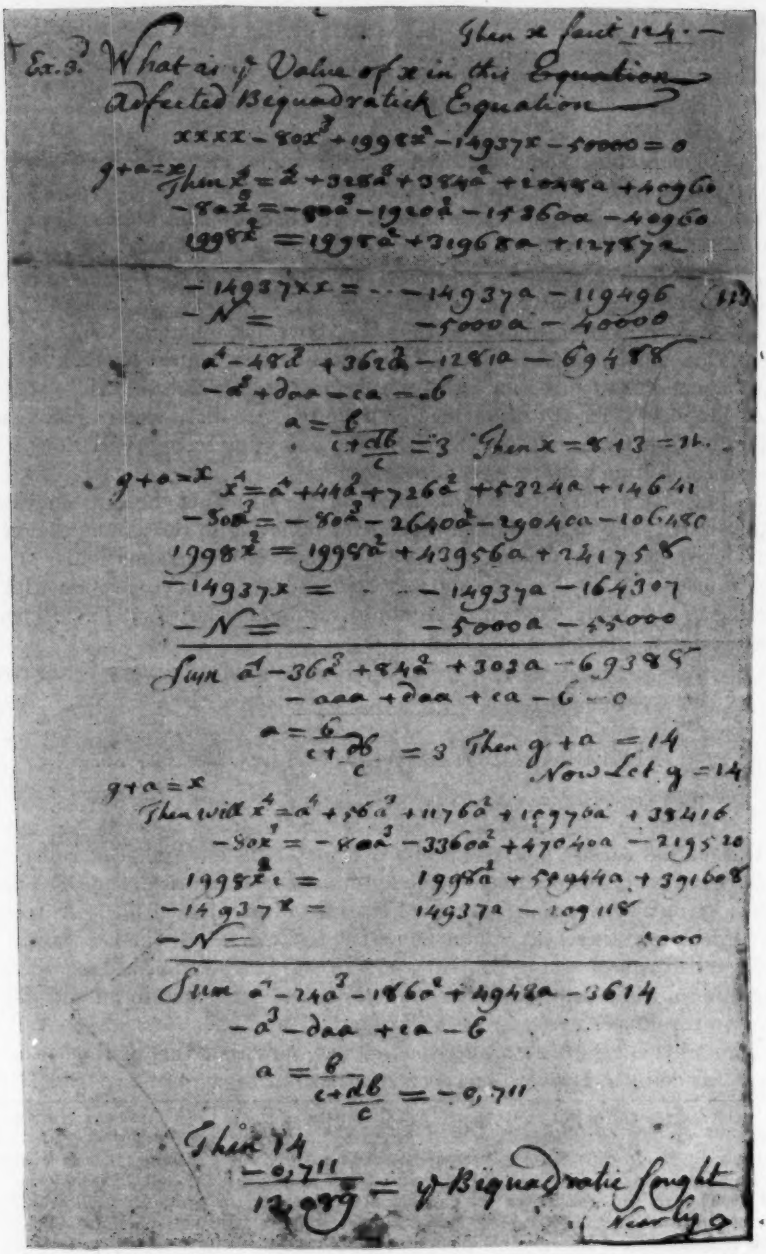
ALGEBRA AT HARVARD COLLEGE IN 1730.

By LAO GENEVRA SIMONS, Hunter College of the City of New York.

One of the earliest professors of mathematics in an American college was Isaac Greenwood who occupied the chair founded at Harvard College by Thomas Hollis. Greenwood was a graduate of Harvard of the class of 1721; he received his A.M. degree three years later and went to England for further study. He was installed as professor in February, 1728. In 1729, he published anonymously a work on arithmetic.[1] His connection with Harvard came to an end in July, 1738, on account of serious irregularities in his own life.

Direct evidence hitherto published of the work that this mathematics professor was doing at Harvard has been so meager as to be quite negligible. Such

[1] Conclusive evidence of the authorship of this arithmetic is to be found in the advertisement which follows: "[Boston] *Weekly News-Letter*, May 29, 1729. Just Published. Arithmetick Vulgar & Decimal; with the Application thereof to a Variety of Cases in Trade & Commerce. By Isaac Greenwood, A.M., Hollisian Professor of Mathematicks, and Philosophy. To be Sold by Thomas Hancock at the Bible & Three Crowns near the Town Dock, Boston." The copy of this work in the New York Public Library contains "Eliakim Willis his Book 1733." Willis graduated from Harvard in 1735 and no doubt used the book in his college course.



Solution of a biquadratic equation from the manuscript of a Harvard student, 1730.

evidence as there is probably passed under the eyes of a great many persons until one looked upon it who knew its worth. The discovery of a complete manuscript notebook on algebra dated 1739 greatly enriched our knowledge of the beginnings of algebra in America.[1] The manuscript referred to was found in the museum housed in the old jail in York Village, Maine. It is by Samuel Langdon who graduated from Harvard in 1740 and some years later became president of the College.

The value of this manuscript was considerably enhanced by the further discovery, in the *Manuscripts Americana* at Harvard University, of another manuscript which so closely resembles the Langdon one as to leave no doubt that the two notebooks were taken from the same original. The Harvard manuscript was written by James Diman, who graduated from Harvard in 1730, and who served the College as librarian from 1735–1737.

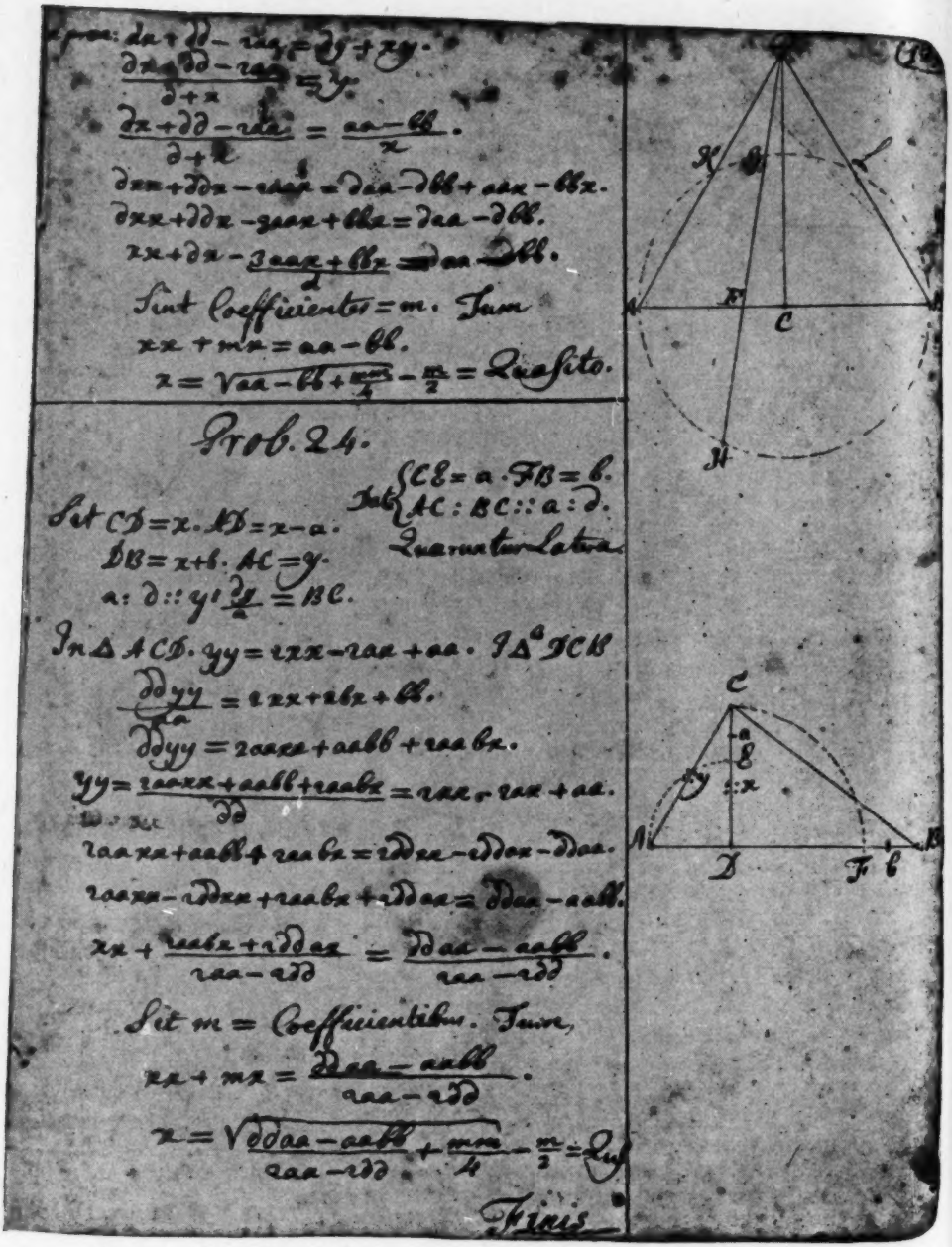
The Langdon manuscript consists of seventy-five numbered pages, with two unnumbered pages forming a front cover and eighteen unnumbered pages at the end. The latter contain no notes, except one leaf. The pages are 14.7 cm. by 18.7 cm., and the book consists of forty-eight leaves, three-fourths of one leaf having been cut out by the author. On the front cover appears the inscription: "Samuel Langdon's Book, July 25, 1739," and on the reverse of this leaf are the words: "Algebra by Isaac Greenwood, M.A. Began July 25, 1739." A colophon reads: "Finished writing Algebra August 17, 1739. Algebrae Finis."

The Diman manuscript consists of four unnumbered pages followed by one hundred and twenty-five numbered pages, 16 cm. by 19.3 cm. In the upper right-hand corner of the first page is the inscription: "James Diman's Book 1730/31." In large writing on this same page appears the title: "Algebra or Universal Mathematics reviewed 1738 with Notes and Additions." The third page has the following note: "Books perused in y^{e} review of my Algebra made in 1738. 1. Harris Lexicon Technicum. 2. Chambers Cyclopaedia. 3. Wolfius Elementa Matheseos Univers." The work ends on page 125 simply with "Finis."

The earlier date, 1730–31, on the Diman manuscript occurred during Greenwood's first years in his professorship, and hence there is every reason to believe that Greenwood was using this algebra material at least as early as 1730. The other date, August 1738, on this same manuscript was just after Greenwood's withdrawal from the College. A comparison of the two shows just a small section that is not found in the Langdon one and so Diman added little to the original in his review. Langdon's book was made about a year later in July 1739. Both manuscripts seem to be careful copies of work done earlier at Harvard.

The existence of two such manuscripts is a matter of importance in the history of American mathematics. One manuscript might have been the work given to a private pupil, but two similar manuscripts at different times during the same professorship afford unmistakable proof that such work was being taught at Harvard during the period of this particular man.

[1] David Eugene Smith, "A Glimpse at Early Colonial Algebra." *School and Society*, January 5, 1918: This article fully describes the manuscript and any reference to it herein contained will necessarily cover the same ground, although it is the result of a study made by the writer.



Algebra applied to the solution of geometric problems from the Diman (1730) manuscript.

Some discussion of the subject matter of these notebooks will show the extent of the algebra presented by the first professor at Harvard. The introduction denotes an interest on the part of the author of the original in the history of the subject. It begins as follows:

This science is called Algebra from two or three words en y^{e} Arabian Language, w^{ch} may be interpreted either the Art of Restitution, & Comparison; or y^{e} Art of Resolution & Equation. It is also known by various other Names. The first y^{e} wrote upon this Subject, in Europe, termed it y^{e} Rule of Restitution & Opposition; Since, it has been called by some, the Analytick Art; by others, Specious Computation; Regula Rei et Census; y^{e} great Art; Modern Geometry; Universal Mathematicks &c.[1]

The topics treated in the notebooks are almost identical except for variations in spelling. Given in the words of the Diman manuscript, they are:

Notation, Algebraical Caracters, Addition of Integers, Subtraction, [Langdon uses 'Substraction'[2]], Multiplication of Algebraic Integers, Division, Algebraical Fractions, Addition and Subtraction of Fractions, Multiplication of Fractions, Division of Fractions, Involution of Whole Quantities, Involution of Fractional Quantities, Evolution of Whole Quantities, Fractional Evolution, Binomial Quantities, Involution [Of Binomial Quantities], Promiscuous Examples, Multinomial Quantities, Involution [Of Multinomial Compound Quantities], Evolution, Surd Quantities, Notation [Of Surds], Reduction of Surds, Multiplication of Surds, Division of Surds, Addition and Subtraction of Surds, Compound Surds, Multiplication of Binomial Surds, Division in Compound Surds, ['Compound Surds' is the one topic in which there is a marked difference in the two manuscripts, the Langdon one containing much less material than the Diman], Equation, Reduction of Equations, Reduction by Addition, Reduction by Subtraction, Reduction by Multiplication, Reduction by Division, Reduction by Involution, Reduction by Evolution, Reduction by Analogies to Equations & e Contra, The Method of Resolving Algebraical Questions, General Rules concerning y^{e} Reduction of Equations, Simple Equations, The Solution of Adfected Quadratick Equations, M^{r}. Oughtreds method of solving adfected Quadraticks, The Solution of Adfected Equations by taking away y^{e} Second Term,[3] The Solution of Adfected Quadratick Equations by y^{e} method of Compleating y^{e} Square, Questions, The Resolution of Cubic Equations, Cubic Equations by Substitution, Cubic Equations by Tryalls and Depression, The Solution of Irregular Cubics, The Method of Converging Series, M^{r}. Raphson's Theorems for Simple Powers, M^{r}. Raphson's Theorems for Adfected Equations, D^{r}. Halley's Theorems for Solving Equations of all sorts, Concerning the Method of resolving Geometrical Problems algebraically.

A few interesting passages will be given to indicate the spirit of these textbooks.

Involution of Binomial Quantities. . . . Consequently, if y^{e} Numeral figures of Coefficients could be found y^{e} whole might be performed without multiplication and this is done by y^{e} following problem. [Langdon says "This is done by y^{e} following rule, given by S^{r}. Isaac Newton, see p. 139."[4]]. To find y^{e} Coefficients in Binomial Powers. Rule. Multiply y^{e} Coefficient into y^{e} Index of y^{e} Power and Divide that Product by y^{e} Number of terms, counting from y^{e} left hand, and y^{e} Quotient will be y^{e} Coefficient or Numeral Figure of y^{e} next successive Quantity.

Irrational Quantities are noted thus $\sqrt{\cdot}:2$ w^{ch} is 2 wth y^{e} Sign of Irrationality $\sqrt{\cdot}$ before it. . . . There is also another way of marking surd Quantities where Roots are expressed without y^{e}

[1] Quotations are from the earlier manuscript, that by Diman; citations from the Langdon one would be practically the same. The latter appeared in: David Eugene Smith, "A Glimpse at Early Colonial Algebra," *loc. cit.*

[2] For a discussion of the origin of this word, see David Eugene Smith, *History of Mathematics*, II, 95, Boston, 1924.

[3] In this method $y - \frac{1}{2}d$ is substituted in the equation $x^2 + dx = m$; whence $y^2 - yd + \frac{1}{4}d^2 = x^2$ and $y^2 = m + \frac{1}{4}d^2$ or $y = \sqrt{m + \frac{1}{4}d^2}$.

[4] The work referred to is entitled: *The Elements of that Mathematical Art commonly called Algebra* by John Kersey, London, 1673.

Radical sign by their Index, this is founded upon y^{e} manner of expressing Powers, thus x^2, x^3, x^4, signifie y^{e} Square, Cube & Biquadratick of x: so $x\frac{1}{2}$, $x\frac{1}{3}$, $x\frac{1}{4}$ will accordingly signifie y^{e} Square, Cube and Biquadratick Root of x. . . . and w^{n} at any time there is y^{e} Sign of Irrationality prefixt to mixt Quantities with y^{e} sign of Inseperation over y^{m} thus $\sqrt{:}\ ^{3}\overline{7 + \sqrt{:}\,2}$ it is called a Universal Root.

Reduction of Analogies to Equations & e Contra.
Reduce y^{e} Analogy $x : 4 : 2x$, $4 \times 4 = 16$, $x \times 2x = 2xx$,

$$\frac{2xx}{2} = \frac{16}{2} = 8 \text{ per } 16 : 6 \text{ Euclid, } xx = 8.]$$

Sets of problems follow the rules for the solution of simple and quadratic equations. It is of especial interest to note that in connection with the solution of question 7 of the set under quadratic equations, the Langdon manuscript (p. 49) gives an imaginary number in the result. This is the only approach to an imaginary in either of the manuscripts, and it indicates scholarship on the part of the author of the notes as well as ability on the part of his young pupil.

The advanced nature of the work done at Harvard in 1730 in the subject of algebra is indicated in the treatment of cubic and higher degree equations. Three methods are given for the solution of cubic equations. The first is that "By Substitution, Deduction & Division." The second method is by "Tryalls and Depression," and the author tells that "The Resolution of Cubic Equations by Tryalls may be performed two ways." The first way suggests the beginning of the application of the Remainder Theorem, since divisors of the absolute term, which must always be transposed to the first member of the equation, are sought out. Then the cubic expression is to be divided by $x +$ or $x -$ this divisor. The quotient will be a quadratic expression which may be treated as such or the above process may be repeated. The other way again takes factors of the absolute term and substitutes these factors for the unknown, one by one, until one is found which makes the first member of the equation amount to the absolute term.

With the third method, "The Method of Converging Series," the more difficult handling of cubic equations appears. As defined in the book:

> The Method of Converging Series is an Approximation, or orderly approach nearer & nearer y^{e} Truth y^{e} more one works on ad Infinitum, by w^{ch} means any Equation w^{t}soever either Quadratick, Cubic, Biquadratick, Sursolid, etc. may be answer'd to any degree of exactness.

After some further general remarks on this method, there follow "M^{r}. Raphson's[1] Theorems for Simple Powers." "For y^{e} Biquadratick $a^4 = n - (n - g^4)/4ggg = x$." It is applied "To extract y^{e} Biquadrate of 90."

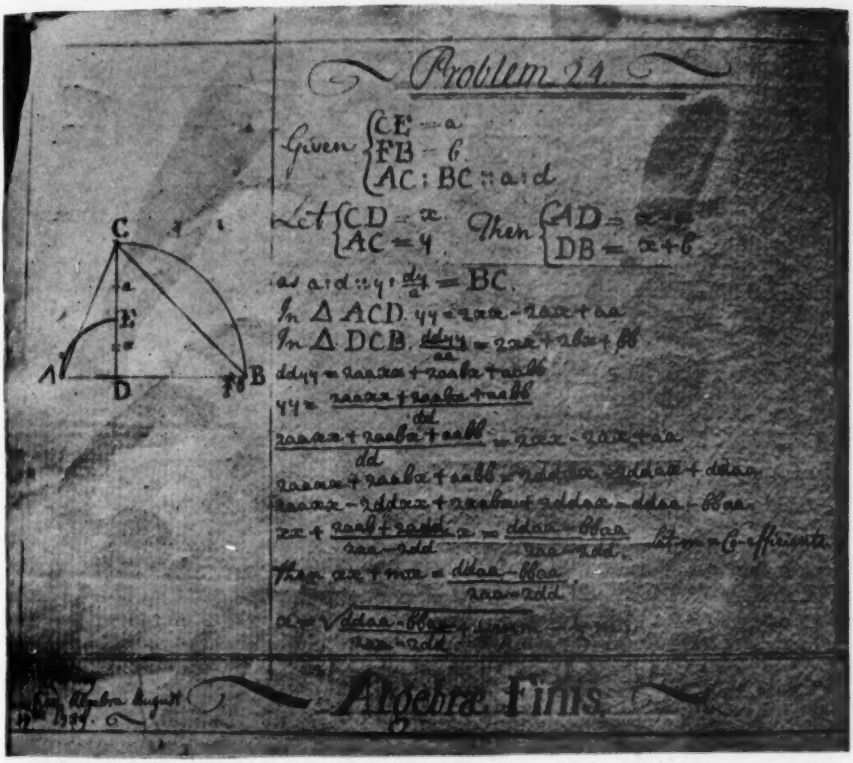
In addition to the theorems for simple powers, there are "M^{r}. Raphson's Theorems for adfected Equations." For the solution of the "Biquadrate $a^4 \pm pa^3 = N$, Theorem $(N - g^4 \mp pg^3)/(4g^3 \pm 3pg^2) = x$" and the theorems for the equations of higher degrees are correspondingly difficult.

[1] Joseph Raphson, who published in 1690 a work entitled *Analysis aequationum universalis*, in which he modified Newton's method of finding the approximate roots of a numerical equation. For discussion, see F. Cajori, *Oughtred*, p. 140.

The last section of this part of the work is devoted to "Dr. Halley's [1] Theorems for Solving Equations of all sorts." As the text indicates:

Dr. Halley has found out two universal Theorems which answer all cases Imaginable of any parts whatsoever, wth greater ease and expedition yn yn others.

The theorems are applied to the solution of several equations.



Algebra applied to the solution of a geometric problem from the Langdon (1739) manuscript.

The notebooks close with a section "Concerning the Method of resolving Geometrical Problems Algebraically," without which no algebra textbook of the eighteenth century was complete.

It is one of the significant topics of these two manuscripts.

In the opening paragraph, the author writes:

But in Geometrical Problems is tho't sufficient to note only such Particulars as are necessary to lead yn Geometritian to some known Theorem, whereby yn solution may be made. And to facilitate this I must advise yn Student, always in Geometrical affairs, to consider yn unknown

[1] Edmund Halley, the great astronomer. He was also deeply interested in algebra and geometry.

Quantity as really known. Then Compairing y• several Quantities in y• Problem, note how they are related either directly or by Consequence of any of Euclid, &c. Demonstrations.

Twenty-four problems of a geometric nature follow. Diman uses Latin throughout this set, while Langdon continues to use the English language. The problems are the same and are numbered alike in the two manuscripts. No statement of the problem in words precedes the solution, except in one instance. The diagram in each case is clearly marked, and the data given by reference to the diagram. There are scattered throughout the exercises many facts from Euclid, both definitely referred to and implied. We cannot doubt that the pupils who studied this algebra, writing out these manuscripts as textbooks, had already taken a good course in geometry.

The two manuscripts under discussion bear a close resemblance to each other, but they differ enough to indicate that the individualities of the students themselves played a part in the final productions. They are both unquestionably based on a set of notes prepared by Professor Greenwood to use in a course at Harvard. Had he lived out his professional life, it is altogether probable that we should have had from his pen the first printed work on algebra written by a native American. Work of the character shown in these manuscripts is evidence that mathematics of a collegiate nature measured by present day standards was taught to students at Harvard College in 1730.

ON THE CORRELATION BETWEEN TWO VARIATES x AND $y = kx^n$.

By KARL PEARSON, University of London, England.

In a paper recently published in the MONTHLY (*1924*, 227–231), Professor P. R. Rider discusses the above problem in the special case of the normal distribution of the variate x. As a practical statistician I should like to consider the problem from a rather more general standpoint with simpler analysis.

Let $\mu_s' = S(x^s)/N$ be the sth moment coefficient about any fixed origin of the character x in a population of N, the symbol S denoting summation for all individual values. Then: $\mu_1' = S(x)/N =$ *mean value of* $x = \bar{x}$, say. Let μ_s be the moment sth coefficient about the mean. Then it follows from a simple application of the binomial theorem that

$$\mu_s' = \mu_s + s\bar{x}\mu_{s-1} + \frac{s(s-1)}{1\cdot 2}\bar{x}^2\mu_{s-2} + \text{etc.} \qquad (i)$$

If r be the correlation coefficient of x and y,

$$r = \frac{S(kx^{s+1}) - \bar{x}kS(x^s)}{N\sigma_x\sigma_y},$$

where σ_x = standard deviation of $x = \sqrt{\mu_2}$, while

$$\sigma_y^2 = \frac{S(k^2x^{2s}) - k^2[S(x^s)]^2/N}{N},$$

and accordingly

$$\sigma_y^2 = k^2(\mu_{2s}' - \mu_s'^2) \qquad (ii)$$

and

$$r = \frac{\mu_{s+1}' - \bar{x}\mu_s'}{\sqrt{\mu_2(\mu_{2s}' - \mu_s'^2)}}. \qquad (iii)$$

This is the simple expression for r. By aid of (i) we can always convert the moments about the fixed origin into moments about the mean.

Thus:

$$r = \frac{\mu_{s+1} + s\bar{x}\mu_s + \frac{(s+1)s}{1\cdot 2}\bar{x}^2\mu_{s-1} + \cdots - \bar{x}(s\bar{x}\mu_{s-1} + \frac{s(s-1)}{1\cdot 2}\bar{x}^2\mu_{s-2} + \cdots)}{\left\{\mu_2\left(\mu_{2s} + 2s\bar{x}\mu_{2s-1} + \frac{2s(2s-1)}{1\cdot 2}\bar{x}^2\mu_{2s-2} + \cdots - (\mu_s + s\bar{x}\mu_{s-1} + \frac{s(s-1)}{1\cdot 2}\bar{x}^2\mu_{s-2} + \cdots\right)^2\right\}^{1/2}}. \qquad (iv)$$

Illustrations:

(a) $s = 1$:

$$r = \frac{\mu_2 + \bar{x}^2 - \bar{x}^2}{\{\mu_2(\mu_2 + \bar{x}^2 - \bar{x}^2)\}^{1/2}} = 1,$$

since $\mu_1 = 0$ and $\mu_0 = 1$.

There is therefore perfect correlation between x and kx, as might be anticipated.

(b) $s = 2$:

Let us write as usual $\sqrt{\beta_1} = \mu^3/\mu_2^{3/2}$ and $\beta_2 = \mu_4/\mu_2^2$. Then applying (iv) we find at once:

$$r = \frac{\frac{\bar{x}}{\sigma_x} + \frac{1}{2}\sqrt{\beta_1}}{\sqrt{\left(\frac{\bar{x}}{\sigma_x} + \frac{1}{2}\sqrt{\beta_1}\right)^2 + \frac{1}{4}(\beta_2 - \beta_1 - 1)}}.$$

As β_2 is always $\geqq \beta_1 + 1$, this is always less than unity. Expanding:

$$r = 1 - \tfrac{1}{8}(\beta_2 - \beta_1 - 1)\frac{1}{\left(\frac{\bar{x}}{\sigma_x} + \frac{1}{2}\sqrt{\beta_1}\right)^2} + \text{etc.}$$

Now $\bar{x}/\sigma_x = 100/V_x$, where V_x is the so-called coefficient of variation. Now in the usual anthropometric characters in man V_x is of the order 3 to 5, and though exceptions can of course be found, this is much the order of variation in plant and animal life. Accordingly, the important term is of the order 1/1000 for the sort of value of β_1 and β_2 usually met with, or $r = .999$.

(c) $s = 3$:

Let us put $\beta_3' = \mu_s/\sigma_x^5$, $\beta_4 = \mu_6/\mu_2^3$, then we have

$$r = \frac{\left(\frac{\bar{x}}{\sigma_x}\right)^2 + \sqrt{\beta_1}\frac{\bar{x}}{\sigma_x} + \frac{1}{3}\beta_2}{D},$$

where

$$D = \left\{\left(\left(\frac{\bar{x}}{\sigma_x}\right)^2 + \sqrt{\beta_1}\frac{\bar{x}}{\sigma_x} + \tfrac{1}{3}\beta_2\right)^2 + \frac{\bar{x}^2}{\sigma x^2}(\beta_2 - \beta_1 - 1) + \tfrac{2}{3}\frac{\bar{x}}{\sigma_x}[\beta_3' - \sqrt{\beta_1}(1 + \beta_2)] + \tfrac{1}{9}(\beta_4 - \beta_2 - \beta_1)\right\}^{1/2}.$$

The value of r as thus determined is nearly equivalent to the following:

$$r = 1 - \tfrac{1}{2}\frac{\frac{\bar{x}^2}{\sigma_x^2}(\beta_2 - \beta_1 - 1) + \frac{2}{3}\frac{\bar{x}}{\sigma_x}[\beta_3' - \sqrt{\beta_1}(1 + \beta_2)] + \frac{1}{9}(\beta_4 - \beta_2 - \beta_1)}{\left\{\left(\frac{\bar{x}}{\sigma_x}\right)^2 + \sqrt{\beta_1}\frac{\bar{x}}{\sigma_x} + \tfrac{1}{3}\beta_2\right\}^2}.$$

To appreciate the order of the deviation from unity in most practical statistical work, let us apply the result to normal distribution for x. We have $\beta_2 = 3$, $\beta_4 = 15$, $\beta_1 = \beta_3' = 0$ and

$$r = 1 - \frac{\frac{\bar{x}^2}{\sigma_x^2} + \frac{2}{3}}{\left(\frac{\bar{x}^2}{\sigma_x^2} + 1\right)^2} = 1 - \frac{1}{\frac{\bar{x}^2}{\sigma_x^2} + 1}, \text{ nearly,} \qquad = 1 - \frac{1}{\frac{(100)^2}{V_x^2} + 1}, \text{ nearly.}$$

If V_x be of the order 4,

$$r = 1 - \frac{1}{625} = .9984,$$

which is perfect correlation for all practical purposes.

Long ago [1] I showed that nothing was gained practically in intensity of correlation by correlating intelligence with weight per stature squared or cubed in the case of adults instead of with weight per simple stature, and those scientists who invent indices of (weight)/(stature)^n as practical measures are wasting their energies. It is of course true that the correlation *coefficient* of x and kx^s is not unity. But it is a very familiar result that r is not unity for perfect correlation unless the regression is *linear*. In all cases of skew regression where the correlation is perfect, as in the case of x and kx^s, what is unity is the correlation ratio y, and no statistician of today lays any stress on the correlation coefficient until he has satisfied himself by computing $\eta_{x \cdot y}$ and $\eta_{y \cdot x}$ and seeing that they do not differ sensibly from r, that the regression is for the range of his problem linear. Then, and then only, r is of practical significance. When, however, the coefficient of variation is of the order 3 to 6, then the portion of the regression curve $y = kx^s$ which actually comes into consideration is sensibly straight and r will not differ from unity by more than 1 or 2 in 1000. Even for considerably higher values of V, r will not differ from unity by more than 1 per cent., which is adequate approximation to unity for most statistical treatment.

[1] *Proc. Royal Soc.*, vol. 71 (1902), p. 111.

There is a good deal of idle talk about a second character failing to follow a normal distribution if it be a function of a first character which does. There is nothing in this criticism, if the coefficient of variation does not take unusual values. If it does take unusual values, then the statistician has been warned of this because his correlation ratios differ sensibly from his correlation coefficient. He then knows that his regression is non-linear and therefore that no normal Gaussian surface can describe the frequency distribution, and accordingly the marginal totals, one or both of them, cannot be normal.

But the deviation from normality is usually very small, the frequency curve to a first approximation being:

$$z = z_0 e^{-\frac{1}{2}\frac{y^2}{\sigma_y^2}}\left[1 + \left(\frac{1}{s} - 1\right) y/\bar{y}\right],$$

where the origin is at the mean of $\bar{y}$. This may be written

$$z = z_0' e^{-\frac{1}{2}\left(\frac{y}{\sigma_y} + \frac{s-1}{s} v_y\right)^2},$$

where z_0' is another constant. We have therefore, still to the same approximation, the curve of frequency, a normal curve, but with its origin not at its mean, but shifted through a distance $-\dfrac{(s-1)}{s} v_y \sigma_y$ or

$$-\frac{(s-1)}{s} v_y^2 \bar{y} = -s(s-1) v_x^2 \bar{y},$$

since $v_y^2 = s^2 v_x^2$ closely.

If s be not greater than 3, then for a value of a coefficient of variation 4—*i.e.*, $V_x = 100 v_x = 4$—this shifting is only through 1 per cent. of $\bar{y}$. If s be fractional, it is far less. Of course there are occasionally higher values of V_x, but the practical statistician will always protect himself against error if he compares his correlation ratios with his correlation coefficient, and bears in mind the magnitude of his coefficients of variation. In actual data I should not be at all surprised if the weights of apples gave quite as good (or bad) a fit to a normal frequency distribution curve as their diameters. There are so many other factors of variation that such a mathematical relation as: weight varies as the cube of the diameter, plays but a small part in the total effect.

NOTE ON THE GEOMETRIC ASPECTS OF EINSTEIN'S THEORY.

By C. N. REYNOLDS, West Virginia University.

A most valuable feature of Professor Pierpont's recent exposition [1] of the geometric aspects of the theory of relativity was the inclusion of a sequence of examples illustrative of the various sections of his paper. These will be of value to many who have occasion to teach elementary courses in the theory of relativity. In my belief that the best approach to this subject is through the two-dimensional case, I suggest the consideration of the following additional exercises, throughout which we shall make use of Pierpont's notation and take $n = 2$.

*A*1. Evaluate (12, 12) explicitly.

*A*2. Prove $aG_{\alpha\mu} = -a_{\alpha\mu}(12, 12), \quad \alpha, \mu = 1, 2.$

*A*3. Prove $G = -\frac{2}{a}(12, 12).$

*A*4. Prove $G_{ij} - \frac{1}{2}a_{ij}G = 0, \quad i, j = 1, 2.$

If H is invariant and $H_{\sigma\tau} = Ha_{\sigma\tau}$, then

*A*5. $H_{\sigma\tau|\nu} = a_{\sigma\tau}\dfrac{\partial H}{\partial x_\nu}$; div $H_{\sigma\tau} = \sum_\tau a_{\tau\tau}\dfrac{\partial H}{\partial x_\tau}$; and div $H_\tau^\rho = \dfrac{\partial H}{\partial x_\tau}$.

*A*6. If $a \neq 0$, then the vanishing of either of the above divergences implies that H is constant.

Replacing equation (86) of Pierpont's paper by

$$T^{\rho\tau} = \rho\frac{dx_\rho}{ds}\frac{dx_\tau}{ds} - pa^{\rho\tau}, \quad \text{or} \quad T_{\rho\sigma} = \rho a_{\alpha\sigma}a_{\rho\tau}\frac{dx_\alpha}{ds}\frac{dx_\tau}{ds} - pa_{\rho\sigma},$$

where p is the hydrostatic pressure,[2] we have, from physical considerations,

$$\text{div } T^{\rho\tau} = \text{div } T_{\rho\sigma} = 0.$$

*B*1. Prove $T = \rho - 2p$; and $T_{\rho\sigma}\dfrac{dx_\rho}{ds}\dfrac{dx_\sigma}{ds} = \rho - p.$

*B*2. Show that if $T_{\rho\sigma}$ be set equal to a linear combination of metric covariants of the second order of the form $aGa_{\rho\sigma} + bG_{\rho\sigma} + ca_{\rho\sigma}$ and if $\lambda \equiv \left(a + \dfrac{b}{2}\right)G + c$, where a, b, and c are constants, then

(*a*) $T_{\rho\sigma} = \lambda a_{\rho\sigma}$.

(*b*) λ is constant and equal to minus p.

(*c*) our metric is that of a spherical surface of radius $\sqrt{\dfrac{2a + b}{c - \lambda}}$.

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[1] *Annals of Mathematics*, 1923, Series II, vol. 23, p. 228.

[2] *Cf.* Birkhoff: *Relativity and Modern Physics*, chap. VII.

Turning now to the case of a static field, we set

$$x_1 = r, \qquad x_2 = t, \qquad \alpha_{12} = 0, \qquad \frac{\partial a_{11}}{\partial t} = \frac{\partial a_{22}}{\partial t} = 0.$$

Since $ds^2 = 0$ for a light impulse, we shall set $a_{22} = -C^2(r)a_{11}$, where $C(r)$ is the observed velocity of light.

*C*1. The equations of a geodesic are equivalent to

$$\begin{cases} a_{22}\dfrac{dt}{ds} = A, \\ a_{22}\left[-C^{-2}\left(\dfrac{dr}{ds}\right)^2 + \left(\dfrac{dt}{ds}\right)^2\right] = 1, \end{cases}$$

where A is a constant of integration.

*C*2. The four equations $G_{ij} = 0$ $(i, j = 1, 2)$ are equivalent to the single equation $(12, 12) = 0$ when $a \neq 0$. Integrating this equation, we have

$$a_{22} = k^2 e^{\pm 2c\int \frac{dr}{C(r)}}.$$

*C*3. The equations of a geodesic imply that $\dfrac{dr}{dt} = C(r)\tanh c(t - t_0)$, whence $a_{22} = A^2 \operatorname{sech}^2 c(t - t_0)$.

*C*4. Show that our metric is that of a right circular cylinder of radius $\dfrac{2A}{c}$. Express the cylindrical coördinates in terms of the observed velocity of light, $C(r)$.[1]

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

The department of Questions and Discussions in the MONTHLY is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems (especially new problems) which are reserved for the separate department of Problems and Solutions.

REPLIES TO QUESTIONS.

52. Is there any simple treatment of the regular pentagon constructions the proof of which involves neither medial section nor trigonometrical formulæ?

I. REPLY BY NORMAN ANNING, University of Michigan.

1. Introduction. In order to construct a regular pentagon without explicit use of medial section or trigonometry, it will be necessary to invoke the aid of

[1] At Professor Pierpont's suggestion I take this opportunity to point out a serious misprint which he found in line 11, page 250, of his paper in volume 23 of the *Annals of Mathematics* to which reference has been made. This line now reads: "by the vanishing of the energy-momentum tensor T," whereas it should read: "by the vanishing of the *divergence* of the energy-momentum tensor T," as is clearly shown by the context.

theorems in which the trigonometry is latent. This note employs Ptolemy's theorem and a moment-of-inertia theorem neither of which requires for its proof any proposition outside of the first six books of Euclid.

2. Analysis. Suppose that a regular decagon $AD'BE'CA'DB'EC'$ has been constructed in a circle whose center is O and whose radius is r. Let $A'B = x$ and $A'C = y$.

The centroid of unit masses at the vertices of the regular pentagon $ABCDE$ is O. For the moment of inertia about an axis through A' perpendicular to the plane of the pentagon, we have

$$\Sigma(AA')^2 = \Sigma AO^2 + 5(OA')^2;$$
$$(2r)^2 + x^2 + y^2 + y^2 + x^2 = 5r^2 + 5r^2. \tag{1}$$
$$x^2 + y^2 = 3r^2.$$

Applying Ptolemy's theorem to the inscribed quadrilateral $A'CE'B$,

$$y^2 + xy = CB^2 = AB^2 = 4r^2 - x^2.$$
$$xy + x^2 + y^2 = 4r^2.$$

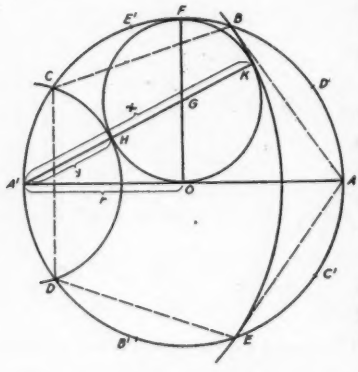
Using (1),

$$xy = r^2. \tag{2}$$

From (1) and (2) by elementary algebra,

$$x - y = r. \tag{3}$$

Two segments x and y which satisfy equations (2) and (3) can be constructed by using any circle in which a chord of length r can be placed. We shall find it convenient to use a circle whose diameter is r.



3. Synthesis. With center O describe a circle of radius r; draw any diameter AOA' and draw OF perpendicular to AA' to meet the circle in F. Draw the circle whose diameter is OF and whose center is G. Draw $A'G$ to meet this latter circle in H and K.

Since $A'K$ is a secant of a circle to which $A'O$ is tangent and since $HK = OF = r$, it follows that $A'K$ and $A'H$ are, respectively, the x and y of the above analysis.

With center A' and radius x describe a circle cutting the original circle in B and E, and, with radius y, one cutting it in C and D. $ABCDE$ is the required regular pentagon.

II. Reply by C. H. Chepmell, Hove, England.

Theorem. (Fig. 1.) *If ABD be a circle (centre C, radius a), and E and F any points in AC and BC respectively, and DF the ordinate at F; then, joining ED,*

and laying off, along AB, EG (= ED), we have[1]

$$a^2 - CG^2 = 2 \cdot EC \cdot FG.$$

For

$$EG^2 = ED^2 = EF^2 + a^2 - CF^2 = EC^2 + 2EC \cdot CF + a^2,$$

also

$$EG^2 = EC^2 + 2EC(CF + FG) + CG^2,$$

and equating the two values of EG^2,

$$EC^2 + 2EC \cdot CF + a^2 = EC^2 + 2EC \cdot CF + 2EC \cdot FG + CG^2$$

or

$$a^2 - CG^2 = 2EC \cdot FG.$$

Q.E.D.

(It will be noted that the magnitude or existence of CF does not affect this result.)

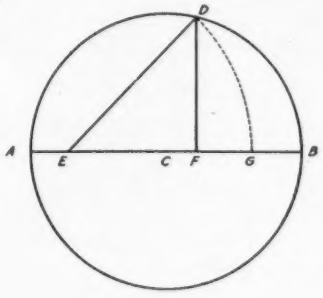


FIG. 1.

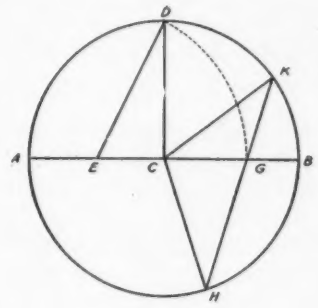


FIG. 2.

Now (Fig. 2), to construct the Pentagon, etc.: if we take E so that $EC = \frac{a}{2}$, and F so as to coincide with C, we have by our theorem above that $a^2 - CG^2 = a \cdot CG$. Draw GH $(= a)$ and produce to K, and join CH, CK. Then $a \cdot GK = HG \cdot GK = AG \cdot GB = a^2 - CG^2 = a \cdot CG$. Hence $GK = CG$; and if we call $\angle KCG$ θ, then $\angle CKG = \theta = \angle CHG$; and $\angle GCH = \angle CGH = 2\theta$; and $5\theta = \pi$.

$$\therefore \quad \angle KCB = 36^\circ \qquad \text{and} \qquad \angle BCH = 72^\circ.$$

This proof has involved no Trigonometry and does not invoke the special relation—"medial section"—but a theorem of much wider and more general application, and capable (see above) of the very simplest proof.

[1] This theorem may be used to solve some problems very neatly; for instance, *Given two sides of a triangle, to construct it so that the angle opposite the greater shall be five times the angle opposite the less.*

DISCUSSIONS.

I. Note on "Limit Proofs in Solid Geometry."

By Paul Capron, United States Naval Academy.

Readers may be interested to learn that Prof. Longley's "Limit Proofs" [*1924*, 196] were for many years taught at the U. S. Naval Academy. How long ago they were introduced into the course, I cannot say, but I think probably about 1865—possibly earlier, probably not later. They were discontinued seven or eight years ago in accordance with a plan to make the course of study at the Naval Academy as much as possible like the conventional courses in the smaller colleges of this country. Some of them were published, but only in texts designed for the use of Midshipmen.

All the proofs were given as Prof. Longley gives them, except that each was worked for a circumscribed set as well as for an inscribed set, and the two sets shown to have a common limit.

II. An Elementary Solution of a Problem of Diophantus.

By A. A. Bennett, University of Texas.

An interesting, elementary, and very ingenious treatment of a problem in Diophantine analysis was given recently by H. E. J. Curzon, in this Monthly.[1] Curzon gives incidentally the solution due to Diophantus, but is engaged chiefly in describing a new solution depending upon three integers, l, m, n, which enter the solutions homogeneously. Their sum is furthermore restricted to be zero, so that we have there essentially only one independent *rational* parameter, although the use of all three parameters makes possible an elegant and symmetrical form for the answer. Three of the solutions that I here offer for the same problem contain two independent rational parameters, and so may be naturally regarded as more extensive than that of Curzon. A new one-parameter solution is also given. To a slight extent necessary as well as sufficient conditions are examined, but the possibility of other two-parameter solutions is not excluded and other independent one-parameter solutions may well exist.

The *form* of the problem may be generalized somewhat from that previously given without altering essentially the character of the solutions. It will be stated as follows. *Given three rational numbers, k_1, k_2, k_3, it is required to find three distinct rational numbers, a, b, c, such that $ab + 2d$, $ab - 2d$, $bc + 2d$, $bc - 2d$, $ca + 2d$, $ca - 2d$ shall all be squares of rational numbers, d being defined as $k_1a + k_2b + k_3c$.*

We shall obtain solutions of the form,

$$\begin{cases} a = cA(s, t), \\ b = cB(s, t), \\ d = c^2D(s, t), \qquad D \text{ not identically zero,} \end{cases}$$

[1] Page 200, June, 1924. This problem is quoted in Carmichael's *Diophantine Analysis* as number 5 of the terminal miscellaneous exercises.

where s and t are arbitrary rational parameters not homogeneously related and where A, B, D are rational functions of s, t, with numerical coefficients. To satisfy the conditions of the problem, we shall also have $d = c(k_1A + k_2B + k_3)$, whence

$$c = (k_1A + k_2B + k_3)/D.$$

Since the coefficients, k, enter nowhere else in the problem, we may dismiss further discussion of them with the remark that in the form in which Diophantus stated the problem these coefficients were all equal to $\frac{1}{2}$, $2d$ being therefore the sum $a + b + c$.

As Curzon notes,[1] it is *sufficient* that rational numbers x, y, z be found such that

$$\begin{cases} bc = x^2 + \dfrac{d^2}{x^2}, \\ ca = y^2 + \dfrac{d^2}{y^2}, \\ ab = z^2 + \dfrac{d^2}{z^2}. \end{cases} \tag{1}$$

We shall now show that this condition is also necessary. By hypothesis, $\sqrt{bc + 2d}$, $\sqrt{bc - 2d}$ are rational, the positive sign being used before the radical to denote a positive number. Then the sum $\sqrt{bc + 2d} + \sqrt{bc - 2d}$ is rational and will be denoted by $2x$. Similarly $2y$ and $2z$ are defined. Now

$$\frac{2d}{x} = \frac{4d}{\sqrt{bc + 2d} + \sqrt{bc - 2d}} = \frac{4d(\sqrt{bc + 2d} - \sqrt{bc - 2d})}{4d} = \sqrt{bc + 2d} - \sqrt{bc - 2d}. \tag{2}$$

Thus

$$x + \frac{d}{x} = \sqrt{bc + 2d}, \quad x - \frac{d}{x} = \sqrt{bc - 2d}.$$

Squaring and adding, we have, as desired, $bc = x^2 + \dfrac{d^2}{x^2}$, and the other relations of (1) follow in like manner.

From the fact that $a = (abc)/(bc)$, we may write

$$a = \frac{P}{\left(x^2 + \dfrac{d^2}{x^2}\right)}, \qquad b = \frac{P}{\left(y^2 + \dfrac{d^2}{y^2}\right)}, \qquad c = \frac{P}{\left(z^2 + \dfrac{d^2}{z^2}\right)}, \tag{3}$$

where P^2 is the product $\left(x^2 + \dfrac{d^2}{x^2}\right)$, $\left(y^2 + \dfrac{d^2}{y^2}\right)$, $\left(z^2 + \dfrac{d^2}{z^2}\right)$. Three possibilities may be imagined. All three, only one, or else none at all of the three

[1] Curzon takes d in the special form of the product, $-xyz$, so that the relations are throughout more specific. All of his work can be carried through with the more general symbol, d, and with perhaps a slight gain in brevity and symmetry but without essential increase in generality.

expressions $\left(x^2+\frac{d^2}{x^2}\right)$, $\left(y^2+\frac{d^2}{y^2}\right)$, $\left(z^2+\frac{d^2}{z^2}\right)$ may be the square of a rational number. Diophantus uses the hypothesis under which each of these is a square. Curzon mentions the example for which $x = 40$, $y = 24$, $z = 15$, $d = 1680$, so that $d/x = 42$, $d/y = 70$, $d/z = 112$. It is impossible that exactly two of these expressions should be squares. We shall not make use of the separate negations in these hypotheses but shall consider only the question of the product of the three being the square of a rational number, thus including all three possibilities.

Curzon's method of finding a solution may be interpreted as leading to the following conclusion. There exists a symmetric function $F(x, y, z)$ such that the product of the three expressions, $x^2+\frac{d^2}{x^2}+F$, $y^2+\frac{d^2}{y^2}+F$, $z^2+\frac{d^2}{z^2}+F$, is the square of a rational function, P, of x, y, z. A choice for this function, F, is found to be given by

$$rF(x, y, z) = d^2p + d(pq - r) + qr,$$

where p, q, r are the symmetric functions $x + y + z$, $xy + yz + zx$, xyz, respectively. This choice gives for P,

$$P = (x + y)(y + z)(z + x)\left(1+\frac{d}{xy}\right)\left(1+\frac{d}{yz}\right)\left(1+\frac{d}{zx}\right).$$

The solution follows as in (3) by so choosing x, y, z that F shall vanish, which suggests the methods of use in multiplicative domains. These remarks upon Curzon's method are given to show for the first time the common background for these two types of solution. The methods here used make use of this common start.

We have in any case

$$\left(x^2+\frac{d^2}{x^2}\right)\left(y^2+\frac{d^2}{y^2}\right) = c^2\left(z^2+\frac{d^2}{z^2}\right). \tag{4}$$

This equation we may write in the form

$$\left(xy+\frac{d^2}{xy}\right)^2 + d^2\left(\frac{x}{y}-\frac{y}{x}\right)^2 = c^2z^2 + d^2\frac{c^2}{z^2}. \tag{5}$$

From this stage on we shall consciously restrict[1] our investigations to such solutions, if any, as satisfy the relations, satisfied by (5) but not necessary to it, namely,

$$\begin{cases} xy+\dfrac{d^2}{xy} = cz, \\ \dfrac{x}{y}-\dfrac{y}{x} = \dfrac{c}{z}. \end{cases} \tag{6}$$

[1] This restriction excludes Curzon's elementary and very ingenious solution.

It may be noted that had the terms been paired in the other order, the only effect would have been to replace z by d/z, and so to yield the same solution for (1). It would have been equally convenient to replace (5) by

$$\left(xy - \frac{d^2}{xy}\right)^2 + d^2\left(\frac{x}{y} + \frac{y}{x}\right)^2 = c^2z^2 + d^2\frac{c^2}{z^2}, \tag{5'}$$

or in (6) to consider other combinations of plus and minus signs for the two right hand members. This is in fact sometimes necessary if we wish to obtain directly answers that are necessarily positive. In practice, however, it is just as well to confine attention to a definite algebraic choice of signs, and then at the end to change all the signs in case that negative numbers are obtained.

From (6) we obtain, by multiplication and removal of factors,

$$\left(1 + \frac{d^2}{x^2y^2}\right)\left(\frac{x^2}{y^2} - 1\right) = \frac{c^2}{y^2}. \tag{7}$$

The left hand member of (7) is then the square of a rational number. Two obvious cases leading to different solutions suggest themselves, from the factored form of the left hand member. Still other solutions might be obtained when the left hand member is expanded and the terms rearranged so as to suggest other devices. One of these will be carried through in (24). For the present we may discuss the alternatives,

(i): $1 + \frac{d^2}{x^2y^2}$ and $\frac{x^2}{y^2} - 1$ are individually squares of rational numbers,

(ii): $1 + \frac{d^2}{x^2y^2} = e\left[\frac{x}{y} - 1\right]$, where e is a rational number, and $e\left[\frac{x}{y} + 1\right]$ is the square of a rational number.

For (i) solutions of (7) are at once obtained in the form

$$\begin{cases} \dfrac{x}{y} = \dfrac{s^2+1}{s^2-1}, & \dfrac{x^2}{y^2} - 1 = \left(\dfrac{2s}{s^2-1}\right)^2, \\ \dfrac{d}{xy} = \dfrac{2t}{t^2-1}, & 1 + \dfrac{d^2}{x^2y^2} = \left(\dfrac{t^2+1}{t^2-1}\right)^2. \end{cases} \tag{8}$$

Thence choosing a particular sign for the square root, we have

$$\frac{t^2+1}{t^2-1} - \frac{2s}{s^2-1} = \frac{c}{y}. \tag{9}$$

From (8) and (9) by elimination, we have

$$\begin{cases} x = c\cdot\dfrac{s^2+1}{2s}\cdot\dfrac{t^2-1}{t^2+1}, \\ y = c\cdot\dfrac{s^2-1}{2s}\cdot\dfrac{t^2-1}{t^2+1}, \\ d = c^2\cdot\dfrac{s^2-1}{2s}\cdot\dfrac{s^2+1}{2s}\cdot\dfrac{2t}{t^2+1}\cdot\dfrac{t^2-1}{t^2+1}. \end{cases} \tag{10}$$

From these and by use of either one of the equations (6), we obtain the same expression for the value of z, namely,

$$z = c \cdot \frac{s^2 - 1}{2s} \cdot \frac{s^2 + 1}{2s}.$$

By substitution in (1), we have finally as the solution under assumption (i), the following:

$$\begin{cases} a = c\left[\left(\dfrac{s^2+1}{2s}\right)^2 - \left(\dfrac{t^2-1}{t^2+1}\right)^2\right] \\ \qquad = c\dfrac{[(s-1)^2t^2 + (s+1)^2][(s+1)^2t^2 + (s-1)^2]}{(2s)^2(t^2+1)^2}, \\ b = c\left[\left(\dfrac{s^2-1}{2s}\right)^2 + \left(\dfrac{t^2-1}{t^2+1}\right)^2\right] \\ \qquad = c\dfrac{[(st-1)^2 + (s-t)^2][(st+1)^2 + (s+t)^2]}{(2s)^2(t^2+1)^2}, \\ d = c^2\left[\dfrac{s^2-1}{2s} \cdot \dfrac{s^2+1}{2s} \cdot \dfrac{2t}{t^2+1} \cdot \dfrac{t^2-1}{t^2+1}\right], \qquad (d \neq 0). \end{cases} \tag{11}$$

For case (ii) we have analogously

$$\begin{cases} e\left(\dfrac{x}{y} - 1\right) = 1 + \dfrac{d^2}{x^2y^2}, \\ e\left(\dfrac{x}{y} + 1\right) = s^2, \end{cases} \tag{12}$$

where s is to be an arbitrary rational parameter. Hence

$$s^2 = \frac{d^2}{x^2y^2} + 1 + 2e.$$

We shall introduce a second arbitrary rational parameter, t, by the condition

$$t = d/(xy). \tag{13}$$

Thus $2e = s^2 - t^2 - 1$, and from (12), $(s^2 - t^2 - 1)\left(\dfrac{x}{y} + 1\right) = 2s^2$ or

$$\frac{x}{y} = \frac{2s^2}{s^2 - t^2 - 1} - 1 = \frac{s^2 + t^2 + 1}{s^2 - t^2 - 1}. \tag{14}$$

Relation (7) now becomes

$$\frac{(2s)^2(1 + t^2)^2}{(s^2 - t^2 - 1)^2} = \frac{c^2}{y^2},$$

whence, choosing a sign for y, we have

$$y = c\frac{s^2 - t^2 - 1}{2s(t^2 + 1)}, \tag{15}$$

From (14) we have therefore

$$x = c\frac{s^2 + t^2 + 1}{2s(t^2 + 1)}. \tag{16}$$

whence by (13) we have

$$d = c^2\frac{1}{(2s)^2}\cdot\frac{t}{(t^2 + 1)^2}\cdot(s^2 - t^2 - 1)(s^2 + t^2 + 1). \tag{17}$$

From either equation in (6) we obtain the same expression for z, namely,

$$z = c\frac{1}{(2s)^2(t^2 + 1)^2}(s^2 - t^2 - 1)(s^2 + t^2 + 1). \tag{18}$$

By substitution in (1), we obtain for the system of solutions in case (ii), the following:

$$\begin{cases} a = c\dfrac{[(s+1)^2 + t^2][(s-1)^2 + t^2]}{(2s)^2(t^2+1)}, \\ b = c\dfrac{[(s+t)^2 + 1][(s-t)^2 + 1]}{(2s)^2(t^2+1)}, \\ d = c^2\dfrac{t[(s^2 - t^2 - 1)(s^2 + t^2 + 1)]}{(2s)^2(t^2+1)^2}, \qquad (d \neq 0). \end{cases} \tag{19}$$

It may be noted that the solutions for (i) and (ii) are shown in closely analogous but noticeably distinct forms. However these two have a common subcase that we may denote by (iii). It will not be easily inferred from the final forms of the solutions but is readily discovered by retracing the steps in the determination of the solution. Thus we may replace, in (i), s by $[s(t^2 - 1)]/(t^2 + 1)$, while leaving t unaltered, or replace, in (ii), t by $(2t)/(t^2 - 1)$, while leaving s unaltered, and obtain the same solution in the two cases. The formulas are further simplified if s is then replaced by $s/(t^2 - 1)$. It is to be noted that this common solution continues to exhibit two independent rational parameters, although not containing all of the solutions of either of the two cases already given. This fact is verified by noting the increase in the degree to which these parameters are raised in the formulas

$$\begin{cases} a = c\dfrac{[(s + t^2 - 1)^2 + 4t^2][(s - t^2 + 1)^2 + 4t^2]}{(2s)^2(t^2+1)^2}, \\ b = c\dfrac{[(s + 2t)^2 + (t^2 - 1)^2][(s - 2t)^2 + (t^2 - 1)^2]}{(2s)^2(t^2+1)^2}, \\ d = c^2\dfrac{2t}{t^2 - 1}\cdot\dfrac{[s^2 - (t^2+1)^2][s^2 + (t^2+1)^2]}{(2s)^2(t^2+1)^2}, \qquad (d \neq 0). \end{cases} \tag{20}$$

Let us examine for a moment an explicit numerical solution for cases (i) and (ii). Since of necessity d does not vanish, we cannot give to s or t the values 0, 1, or -1 in (11). Putting, then, $s = t = 2$, we have from (11)

$$\begin{cases} a = c\dfrac{13\cdot 37}{16\cdot 25} = c\dfrac{481}{400}, \\ b = c\dfrac{9\cdot 41}{16\cdot 25} = c\dfrac{369}{400}, \\ d = c^2\dfrac{9}{20}, \end{cases}$$

whence

$$ac + 2d = c^2\left(\frac{29}{20}\right)^2, \qquad ac - 2d = c^2\left(\frac{11}{30}\right)^2,$$
$$bc + 2d = c^2\left(\frac{27}{20}\right)^2, \qquad bc - 2d = c^2\left(\frac{3}{20}\right)^2,$$
$$ab + 2d = c^2\left(\frac{567}{400}\right)^2, \qquad ab - 2d = c^2\left(\frac{183}{400}\right)^2.$$

If c is to be so determined that $2d = a + b + c$, in accordance with the condition of Diophantus, we shall have

$$c = \frac{125}{36}.$$

Similarly from (19),

$$a = c\frac{13\cdot 5}{16\cdot 5} = c\frac{13}{16}, \qquad b = c\frac{17\cdot 1}{16\cdot 5} = c\frac{17}{80}, \qquad d = c^2\frac{2 - (-1)\cdot 9}{16\cdot 25} = -c^2\frac{9}{200},$$

whence

$$ab + 2d = c^2\left(\frac{23}{80}\right)^2, \qquad ab - 2d = c^2\left(\frac{41}{80}\right)^2,$$
$$bc + 2d = c^2\left(\frac{7}{20}\right)^2, \qquad bc - 2d = c^2\left(\frac{11}{20}\right)^2,$$
$$ca + 2d = c^2\left(\frac{17}{40}\right)^2, \qquad ca - 2d = c^2\left(\frac{19}{20}\right)^2,$$

and if c is to be determined as above,

$$c = -45.$$

The sign of c may be here changed with a resulting change in the numerical representation of a, and b, and without altering the fundamental relations.

Such a change merely corresponds to a different determination in the signs of one or more of the radicals used.

Let us now start afresh from equation (7), writing it in the form

$$\frac{d^2}{y^2} + \left(x^2 - \frac{d^2}{x^2}\right) = c^2 + y^2. \tag{21}$$

For this we seek a solution of the form [1]

$$\frac{d}{y} = c, \qquad x^2 - \frac{d^2}{x^2} = y^2. \tag{22}$$

The second of these reduces to

$$(xy)^2 + d^2 = x^4. \tag{23}$$

This has the family of solutions,

$$xy = 2mn, \qquad d = n^2 - m^2, \qquad x^2 = n^2 + m^2,$$

where $m = n\dfrac{2s}{s^2-1}$, giving

$$x = n\frac{s^2+1}{s^2-1}, \qquad y = n\frac{4s}{s^2+1}, \qquad d = n^2\frac{s^4-6s^2+1}{(s^2-1)^2}.$$

Then from the first equation of (22)

$$n = c\frac{4s(s^2-1)^2}{(s^2+1)(s^4-6s^2+1)},$$

whence

$$\begin{cases} x = c\dfrac{4s(s^2-1)}{s^4-6s^2+1}, \\ y = c\dfrac{16s^2(s^2-1)^2}{(s^2+1)^2(s^4-6s^2+1)}, \\ d = c^2\dfrac{16s^2(s^2-1)^2}{(s^2+1)^2(s^4-6s^2+1)}. \end{cases}$$

Inserting in either equation of (6), we shall have

$$z = c\frac{(s^4-6s^2+1)^2}{4s(s^2-1)(s^2+1)^2}.$$

This gives finally by (1) a new one-parameter solution, which may be directly verified by expansion, namely,

$$\begin{cases} a = c\dfrac{[(s^2-1)^4+(2s)^4]^2+12(s^2-1)^4(2s)^4}{[(s^2-1)^4-(2s)^4]^2}, \\ b = c\dfrac{8(s^2-1)^2(2s)^2[(s^2-1)^4+(2s)^4]}{[(s^2-1)^4-(2s)^4]^2}, \\ d = c^2\dfrac{4(s^2-1)^2(2s)^2}{[(s^2-1)^4-(2s)^4]}. \end{cases} \tag{24}$$

[1] The other pairing fails to give a solution.

RECENT PUBLICATIONS.

EDITED BY W. B. CARVER, Cornell University, to whom books and communications should be sent.

REVIEWS.

S. Lie, *Gesammelte Abhandlungen* herausgegeben von Friedrich Engel und Poul Heegaard; Leipzig, B. G. Teubner; Kristiania, H. Aschehoug & Co., Volumes 3 and 5, 1922 and 1924, 16 + 789 and 12 + 776 pages. Subscription prices 25 and 35 Norwegian crowns.

Niels Henrik Abel (1802–1829) and Marius Sophus Lie (1842–1899) are two of the world's great mathematicians whom Norway has produced. Editions of Abel's collected papers appeared in 1839 and in 1881; indeed for the latter edition, Lie was one of the editors. And now, about twenty-five years after Lie's death, we have two volumes of the fine edition of his collected papers now being published by the Norwegian Mathematical Society with the assistance of grants from the Norwegian Research Fund of 1919, from the Society of Sciences of Christiania, and from the Academy of Sciences at Leipzig [compare *1921*, 318]. It is proposed that the collected papers shall be comprised within seven volumes of which the first two shall be devoted to geometrical works; the third and fourth to memoirs on differential equations; the fifth and sixth to memoirs on groups of transformations; and the seventh to posthumous works. Hence the volumes which have now appeared are the first on differential equations and the first on groups of transformations; for the editing of both, Engel is chiefly responsible.

These volumes are not, by any means, merely reprints of the papers as written; they have been edited with meticulous and satisfying detail. At any point of the new edition of a paper the page of the original source may be at once found; Lie's "Selbstanzeige" of each paper as it appeared in the *Jahrbuch über die Fortschritte der Mathematik* is printed along with the paper; 44 pages are taken to list the corrections made in the originals of the 42 papers (published 1872–83) in volume 3, and of the 24 (published 1874–89) in volume 5; 330 pages are occupied with critical remarks on the papers, and 32 with an account of the beginning of Lie's theory of transformation groups as contained in his correspondence with Adolph Mayer; 38 pages are filled with various indexes, errata lists, etc. The contents of some of these and of later papers will be comparatively familiar to mathematicians through Lie's treatises: *Theorie der Transformationsgruppen* edited by Engel in three large volumes (1888–93), *Vorlesungen über Differentialgleichungen mit bekannten infinitesimal Transformationen* edited by Scheffers (1891), *Vorlesungen über kontinuierliche Gruppen mit geometrischen und anderen Anwendungen* edited by Scheffers (1893), and *Geometrie der Berührungstransformationen* by S. Lie and G. Scheffers of which only the first volume appeared (1896). All of these (except the last two volumes of the work first mentioned) were reviewed in the *Bulletin* of the New York, or American, Mathematical Society.[1]

[1] C. H. Chapman in *Bulletin of the New York Mathematical Society*, vol. 2, pp. 61–71, 1892;

Two English treatises have widened the circle of those familiar with some of Lie's ideas. One of these is J. E. Campbell's *Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups* (Oxford, 1903); the other is America's contribution, namely, A. Cohen's *An Introduction to the Lie Theory of One-Parameter Groups*[1] (Boston, 1911). A brief general account of Lie's work, in English, was given in Klein's *Lectures on Mathematics* (New York, 1894, pp. 9–24) delivered in 1893 at the Evanston Colloquium. (Compare G. A. Miller in this MONTHLY, *1896*, 295–6.) Then E. O. Lovett's English translation of Darboux's article on Lie was published in *Bulletin of the American Mathematical Society* (vol. 5, pp. 367–370, 1899) and in this MONTHLY, *1899*, 99–101, where a portrait, and sketch by G. B. Halsted, are also published (*1899*, 97–98), as well as interesting reminiscences of one of Lie's pupils, G. A. Miller (*1899*, 191–193).

A biographical sketch and an elaborate bibliography (1869–98) by Engel appeared in *Bibliotheca Mathematica* (series 3, vol. 1, 1900, pp. 166–204); and for Max Noether's very valuable memoir one turns to *Mathematische Annalen* (vol. 53, pp. 1–41, 1900). Noether concludes: "Mit ihm schied der originalste und schöpferischste Vertreter der geometrischen Wissenschaft der letzten drei Decennien dieses Jahrhunderts dahin."

Engel's careful editorial work[2] was evidently a labor of love, and mathematicians must be grateful for considerable new light which he has already shed on some of Lie's ideas. Volume 6, the second on transformation groups, is now in the press.

It is to be hoped that the Norwegian Mathematical Society will be encouraged in its undertaking by receiving subscriptions for Lie's collected papers from many libraries and individuals in America. With the present low rate of Norwegian exchange the subscription prices to the volumes make them exceptionally cheap. The regular prices are about one third higher. Subscribers should write to H. Aschehoug & Co., Sehesteds plads 3, Christiania, Norway.

R. C. ARCHIBALD.

Methodik des mathematischen Unterrichts, 3. Teil. Didaktik der angewandten Mathematik. By DR. W. LIETZMANN. Leipzig, Quelle and Meyer, 1924. xii + 234 pages. Price 10 gold marks.

This volume forms part of the seventh volume of the monumental *Handbuch des naturwissenschaftlichen und mathematischen Unterrichts*, edited by Dr. J. Norrenberg, well known as one of the foremost men in educational reform in Prussia. The author, Dr. Lietzmann, is connected with the Reform-Realgymnasium in Göttingen, was the most prominent of the younger German workers in the International Commission on the Teaching of Mathematics, and is editor of one of the leading educational-mathematical periodicals of Europe. The

E. O. Lovett in *Bulletin of the American Mathematical Society*, vol. 4, pp. 155–167, 1898; J. M. Brooks in *B. A. M. S.*, vol. 1, pp. 241–248 (1895); E. O. Lovett in *B. A. M. S.*, vol. 3, pp. 321–350, 1897.

[1] Reviewed by E. J. Wilczynski in *Bulletin of the American Math. Soc.*, vol. 18, pp. 514–5, 1912.

[2] In this Professor Heegaard was doubtless an able assistant.

book is dedicated to Professor Klein and pays just tribute to him for his influence in the reform of mathematical teaching, not only in his native land, but in the world at large; and in the development of the subject this influence appears throughout the book.

The work is a continuation of the one already reviewed in the MONTHLY (*1914*, 252), but this portion is unique in that it approaches the subject of applied mathematics from the pedagogical standpoint, giving the educational problems the serious consideration which has thus far been so rare. Up to the present time the subject has generally been approached from the standpoint of a collector of problems or of a worker in some special field of industry. The book therefore forms a worthy addition to the educational literature of the day—the more worthy because it is written by one who knows the material as well as the pedagogical side of mathematics.

The work is divided into seven chapters, the topics considered being drawing, the question of instruction in industries, geodesy and astronomy, civic economics, mechanics, physics, and philosophy. It contains numerous illustrations, in half-tone and line engraving, showing the models and apparatus used in the best German schools when presenting such topics as central perspective, solid geometry, hydrostatics, geodesy, and astronomy. Dr. Lietzmann has further made his readers his debtors by his indexes—(1) of apparatus, (2) of names and (3) of subjects—and by his convenient although too brief table of biographical notes concerning various contributors to the field of mathematical teaching. The work is made the more valuable by a large number of bibliographical notes, referring largely to the recent literature of the subject. These references are almost entirely to German sources, but the covering of the international field could hardly be expected; indeed, a reader goes to a German book for the German point of view, and it is not an unmixed evil that this point of view is emphasized.

As to the text itself, it will be an inspiration to our American readers. Our high schools confessedly lack the scholarship of those found in Europe, our educational problem being different from that of the older countries—more democratic and less given to an aristocracy of learning. It is a hopeful sign, however, that we are beginning a scientific differentiation of our students, and this will eventually lead to a much higher grade of scholarship in classes composed of the intellectually capable. With this will come the need for such direction as Dr. Lietzmann's work furnishes, and the book will have a wholesome influence upon the present and the coming generation of teachers in this country.

DAVID EUGENE SMITH.

L'analysis situs et la géométrie algébrique. (Collection de monographies sur la théorie des functions publiée sous la direction de M. Emile Borel.) By S. LEFSCHETZ. Paris, Gauthier-Villars and Co., 1924. 154 pages. Price 20 francs.

In line with his numerous articles in various journals, and in particular with his recent prize memoir, Professor Lefschetz here presents a brief but scholarly treatise, fully worthy of the excellent company found in the Borel series.

The first chapter discusses the general properties of analytic varieties including connectivity, orientation, congruences, homologies, cycles, fundamental systems and divisors of zero, without presupposing any previous knowledge of the subject on the part of the reader; then by assuming two fundamental theorems of Poincaré, various models of hyperspatial varieties and cycles are introduced before analyzing the intersection of varieties and of cycles of a fundamental system.

The second chapter is confined to algebraic surfaces; it gives a rapid resumé of linear systems of curves, basis points, and canonical form of the surface, shows that it must be bilateral, and points out an essential difference between birational transformations of surfaces and those of plane curves, on account of the possible appearance on the former of exceptional curves, images of points under the transformation. The cycles of a curve on the surface are studied in detail, then the curve is allowed to describe a linear pencil, the cycles thus describing two-dimensional manifolds. The relations between the cycles are then investigated, and the irregularity of the surface is derived; if p is the genus of the general curve of the pencil and $2(p - q)$ is the rank of the matrix, the elements of which are the number of intersections of fundamental cycles, analogous to the notation of the Severi base, then q is the irregularity.

Chapter three is concerned particularly with the topology of algebraic surfaces, the reduction to a cell by a proper system of cuts, and the criteria that determine when a linear cycle is reducible to an algebraic curve. Then the intersection of linear cycles and those of three dimensions, finally those of two dimensions, are considered from the results of which the formula of Poincaré, generalization of Euler's formula for a polyhedron, is established. After proving a number of important theorems, the procedure is illustrated by the surface which is the image of the pairs of points of two given algebraic curves.

The next chapter, the most significant in the book, applies analysis situs to establish a number of invariants of algebraic surfaces and to point out that it will be possible to present in a single discipline the whole theory of algebraic surfaces as now developed partly by transcendental and partly by geometric methods. The simple and double integrals of the first kind are first defined, then the abelian integrals discussed, regarding y as constant, as given in Picard's *Traité*. This is then compared with Poincaré's method, and the conditions for existence of algebraic curves are derived. The simple integrals (integrals of total differentials) are then taken up; it is shown that the number of independent ones of the first kind is equal to the irregularity of the surface. Linear systems and continuous systems of curves are discussed. It is shown that on a surface having $q = 0$ every complete system is regular. An algebraic two-dimensional cycle is one having an algebraic curve (effective or virtual) for its boundary. The necessary and sufficient condition that a cycle is algebraic is that the periods of the integrals of the first kind associated with it shall be zero. Here the results are again interpreted for the surface image of the pairs of points of two given curves. The geometric genus is the product, and the irregularity is the sum of

the genera of the two curves. These results had been previously found by different methods.

The fifth chapter, algebraic varieties of more than two dimensions, contains a resumé of many of the ideas developed in the prize memoir. This is so recent and so generally accessible that it will not be necessary to consider the chapter further, except that two pretty proofs will be noticed, first the generalization of the Zeuthen-Segre invariant, first established by Alexander, and second, the proof that a complete base of the general surface of order greater than 3 is a plane section (Noether).

The last chapter, analysis situs and abelian functions, begins with the existence theorem, then discusses periodic functions and those with multipliers, like the theta functions, and closes with the determination of the number ρ (Picard-Severi) for abelian varieties.

Then follow twenty-five pages as an appendix, devoted to a discussion of double integrals of the second kind and of simple integrals of the third kind on algebraic surfaces. The idea was not needed in the earlier discussion, but the methods there employed are here extended to the other types for the sake of completeness. The most important contribution is the investigation of the improper double integral of the second kind. A second note gives a description of the Volterra models, with illustrations.

The work is richly provided with references; the treatment is everywhere concise, and considerable maturity is presupposed on the part of the reader, who is supposed to be familiar with the Picard transcendental theory. At times it is also necessary to consult the memoirs of Severi, and to refer to several papers on analysis situs. What the reader does gain is an impression of a curious unity in the new presentation and that no one earlier method is capable of developing this beautiful theory alone.

The book is almost free of typographical errors; the few not found in the short list of errata will cause no confusion to the reader.

VIRGIL SNYDER.

NOTES ON RECENT PUBLICATIONS.

The Biometric Laboratory, University College, London, England, is issuing a twenty-place table of logarithms called *Logarithmetica Britannica*. The computation is being done by Mr. A. J. Thompson and the printing by the Cambridge University Press. The work will appear in nine parts at ten shillings a part and will form, when finished, a quarto volume about two-and-a-half inches thick. Part IX, 90,000 to 100,000, is out now and the remaining parts should appear within the next four years. Professor Karl Pearson says in the preface:

> The success of the undertaking—which it is needless to say is not an enterprise of profit—can only be assured if a large section of the mathematical world shows a readiness to commemorate with us in this fashion the tercentenary of Briggs' *Arithmetica Logarithmica*.

A historical note on Briggs by Professor Pearson will appear in an early number of the MONTHLY.

ARTICLES IN CURRENT PERIODICALS.

The lists appearing regularly in the MONTHLY of articles in current periodicals are intended to include (1) titles of papers in all mathematical journals published in the United States; (2) titles of mathematical papers and reports published by the national and state academies of science and in journals devoted to general science; (3) titles of mathematical papers by American authors published in foreign journals.

EDUCATION, volume 45, no. 2, October, 1924: "Mathematics or hieroglyphics" by A. W. Forbes, 105–106.

JOURNAL OF THE FRANKLIN INSTITUTE, volume 198, no. 5, November, 1924: "Heaviside's operators in engineering and physics" by E. J. Berg, 647–702.

MONIST, volume 34, no. 3, July, 1924: "Number: An introduction to the theory of analytic functions" by G. Mittag-Leffler, 321–357.

SCIENTIFIC MONTHLY, volume 29, no. 5, November, 1924: "Five fundamental concepts of pure mathematics" by G. A. Miller, 496–501.

SCHOOL SCIENCE AND MATHEMATICS, volume 24, no. 7, October, 1924: "Mathematics in the Senior High School" by F. Allen, 716–726.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to H. J. ETTLINGER, 2910 Harris Park Ave., Austin, Texas.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Md.

[*1922*, 178.]

There were two lectures by outside speakers during 1923–24. Dr. H. H. Lloyd of the Goucher chemistry department lectured on the use of mathematics in chemistry. Dr. Harlow Shapley, Director of the Harvard College Observatory, lectured on the "Evolution of the stars." Dr. Shapley's lecture was open to the whole college. A third faculty talk was given by Dr. Clara L. Bacon of our mathematics department on "The mathematical fourth dimension."

The club meets on an average of every two weeks, and the program regularly consists of papers by students. Two of these this year were on "Magic squares" and the "Theory and use of the planimeter."

(Report by Professor F. P. Lewis.)

THE MATHEMATICS CLUB OF CONNECTICUT COLLEGE, New London, Conn.

[*1922*, 24.]

All meetings are open for discussion of topics closely or more remotely connected with mathematics. The paper of the evening furnishes a principal topic. All members are urged to bring in all items of mathematical interest, whether they appear as news items, changes in curricula or entrance requirements, book reviews, jokes, puzzles, or campus talk, and this feature is frequently the backbone of the evening program. Refreshments are served at all meetings.

October 10, 1921. Organization meeting. Election of officers was completed. Officers for the year were: President, Dorothy S. Wheeler '22; Secretary, Marcia Langley '23; Treasurer, Ellen Willcox '24; Chairman of Program Committee, Ruth Wells '24.

November 22, 1921. "The discovery of the decimal system by Stevin" by Marcia Langley '23; "The geometric solution of trigonometric equations" by Dr. David D. Leib.

January 16, 1922. Open meeting. "Some interesting bits of history" by Dr. Leib.

February 20, 1922. Washington's birthday party. Mathematical and other recreations in charge of Ruth Wells '23.

March 21, 1922. "Mathematics in Asia" by Elizabeth Hall '23.

April 19, 1922. "Beginnings of mathematics in Arabia" by Olive Holcombe '23; "Summary of and extracts from 'Religio Mathematici' by D. E. Smith" by Augusta O'Sullivan '23.

May 23, 1922. Annual picnic in Bolleswood. Election of President and Secretary for 1922–1923.

October 4, 1922. Organization meeting. Election of officers was completed. Officers for the year were: President, Marcia Langley '23; Secretary, Olive Holcombe '23; Treasurer, Helena R. Wulf '23; Chairman of Program Committee, Marie Jester '24.

Dr. Leib told of the work and purpose of the Club. Miss Ruth E. Irwin told of the Smith Club.

November 13, 1922. Keyser's *Mathematical Philosophy*. A review by Florence A. Hopkins. All members present then took the Thorndike algebra test.

December 11, 1922. Christmas party at the home of Dr. and Mrs. Leib.

January 8, 1923. "Magic squares" by Ellen Willcox '24; "Theory of numbers" by Florence A. Hopkins '23.

February 13, 1923. Valentine party at the home of Dr. and Mrs. Leib. "Card tricks" by Dorothy McFarland '25; "Some well-known problems" by Helena Wulf '23.

March 12, 1923. "The development of arithmetic" by Sadie Kenig; "Mapmaking—merits and limitations of various types" by Dr. Leib.

April 16, 1923. Annual open meeting. "The fourth dimension—mathematical and philosophical points of view" by Dr. Leib.

May 29, 1923. Annual picnic and election of President for 1923–24.

October 22, 1923. Organization meeting. Election of officers was completed. Officers for the year were: President, Louise Hall '24; Secretary, Marie Jester '24; Treasurer, Verna A. Kelsey '25; Chairman of the Program Committee, Mary Courtney '24.

November 12, 1923. "Mathematical terms in daily life" by Grace D. Byron '24; "Mathematical poems" by Doris E. Miner '24.

December 10, 1923. Christmas party.

January 17, 1923. "The making of magic squares and other arrays" by Jessie Josolowitz '25.

February 11, 1924. "American mathematics and mathematicians" by Elsie Marquardt '24.

March 10, 1924. "Some fallacies in the study of probabilities" by Verna Kelsey '25; "Some troublesome ideas" by Dr. Leib.

April 14, 1924. Open meeting. "Egyptian mathematics" by Professor R. C. Archibald, Brown University.

May 22, 1924. Business meeting. Election of President and Secretary for 1924–25.

May, 1924. Annual picnic. Due to the weather it was held at the home of Dr. Leib.

October 18, 1924. Organization meeting. Officers for the year are: President, Aileen Fowler '25; Secretary, Eleanor Kelly '25; Treasurer, Jessie Josolowitz '25; Chairman of Program Committee, Clarissa Lord '26.

Dr. Leib spoke of the history and purpose of the club and re-read the paper given at the first meeting of the C. C. Mathematics Club on "Dimensionality." Clarissa Lord '26 proposed a number of puzzles, problems and recreations to test the skill and acumen of the club members after the summer vacation.

(Report by Professor Leib.)

GRINNELL COLLEGE MATHEMATICS CLUB, Grinnell, Iowa.

[*1922*, 416.]

November 20, 1923: "Arithmetic," Mr. R. B. McClennon. (In two divisions—historical and problems.) "Practical applications of arithmetic," Mr. Gilbert Pullen.

December 18, 1923: "Geometry," Mr. Dana Corrough. "Zeno's paradoxes," Mr. Edwin Kingery.

January 22, 1924: "Mathematical recreations," Miss Alice Clifton. (Problems of arithmetic, algebra, geometry and chess.) "Paper folding," Miss Marian Piersol.

March 4, 1924: "A problem in projective geometry," Mr. Loren Gray. "Reliability of teacher's marks," Miss Ethel Warnick.

March 18, 1924: "The spring constellations," Miss Margaret Field. "The hyperboloid of one sheet" Mr. Sibert Dave.

April 15, 1924: Problems of arithmetic, algebra, and geometry submitted by members for solution.

The average attendance at the meetings for the year was eleven.

Mathematics Club of the University of Oklahoma, Norman, Okla.

[*1923*, 335.]

The Mathematics Club of the University of Oklahoma was organized on October 26, 1916, by Dr. W. W. Reaves, head of the mathematics department, and a group of students interested in mathematics. The purpose of the organization was to promote interest and stimulate activity in the study of mathematics by offering an opportunity for the presentation and discussion of mathematical subjects of interest and has functioned as such for eight years.

The total membership for the year 1923–24 was thirty-two. The officers were as follows: President, Varnakale Jones '25, Vice-president, Raymond Dragoo '24, Secretary-Treasurer, Mabel Erwin '24.

During the year the following papers were read:
"Applications of mathematics to astronomy" by Dr. J. O. Hassler.
"Magic squares" by Miss Jewel Conkling.
"Magic cubes" by Miss Sara Beth Barbour.
"Flatland and the fourth dimension" by Miss Margaret Rayburn.
"History and use of the abacus" by Miss Ester Billows.
"Theory and applications of the slide rule," by Varnakale Jones.
"How mathematical theory verifies astronomical data" by Dr. J. O. Hassler.
"Different proofs of the Pythagorian proposition" by Miss Mabel Erwin.
"Mathematical fallacies" by Mr. Carl Wilson.
"The great problem of Apollonius" by Dr. N. A. Court.

(Report by Professor Court.)

Mathematical Club of the University of Arkansas, Fayetteville, Ark.

Topics for 1923–24.

"Magic squares," Elmer Nichols.
"What is mathematics?" Assistant Prof. A. D. Campbell.
"The sun and the moon," Dr. A. M. Harding.
"The solution of simultaneous quadratics," Sam Byrd.
"The nine-point circle," Dean G. W. Droke.
"Cycloid and hypocycloid," Marvin Leeper.
"Hyperbolic functions," W. H. Taylor.
"Mathematical wrinkles," Miss Myrtle Farmer.
"The solution of third and fourth degree equations," Bolling Dunn.
"Exponential functions," Miss Emily Heston.
"Application of the binomial theorem to physics," Marshall Hickman.

(Report by Asst. Professor Campbell.)

The Mathematics Club of the University of Maine, Orono, Maine.

[*1923*, 335.]

The following officers were elected for the year 1923–24. President: Vera Savage '24, Vice-president: Ethelyn Percival '24, Secretary-treasurer: Donald Trouant '24, Faculty adviser: Professor Bryan.

The meetings were held as follows:
September 26: "Talk on text books" by Professor N. R. Bryan.
October 10: "Books added to the library" by Ethelyn Percival '24.
November 21: "Who's who in mathematics" by Edwin Hadlock '24 and H. L. Bowen '24.
December 12: "Eclipses of the sun" by Dean J. N. Hart.
January 9: "The meaning of a group in mathematics" by Mr. W. S. Lucas.
February 13: "Periodicals devoted to mathematical subjects" by Mary Copeland '24, Madeline McPhelies '25, and Doris Shorey '26.
February 27: "The mathematical theory of the game of Nim" by Mr. E. C. Brown.

April 23: "The correlation of high school and college work" by Prof. W. S. Evans.
May 14: "The prediction of eclipses" by Mr. W. S. Lucas.

The following were selected for officers for the coming year: President, Donald Trouant '25; Vice-president, Madeline McPhelies '25; Secretary-treasurer, Elizabeth Laughlin '26.

(Report by Elizabeth H. Laughlin.)

The Mathematics Club of The College of the Ozarks, Clarksville, Arkansas.

The Mathematics Club of The College of the Ozarks was organized in October 1921 for the purpose of arousing interest and stimulating activity in the field of mathematics. The present officers of the Club are: president, John Neal '25; vice-pres., Opal Johnson '26; secretaries, Roy Burgess '24, and Annie Moon '26; treasurer, Viola Stegall '27; chaplain, James Boudra '27. Meetings are held monthly in the College Science Hall. Most of the members of the Club are majoring in mathematics and consequently the discussions and reports are mainly mathematical.

Some of our recent discussions, reports and papers were:
"Mathematical philosophy," Roy Burgess '24.
"Introduction to Einstein's theory," John Neal '25.
"History of geometry," Era King '27.
"Pedagogy of mathematics," Annie Moon '26.
"Great contributions to mathematics," Opal Johnson '26.
"Mathematics and religion," James Boudra '27.
"Count Korzybski's concept of man," Ralph Pitts '26.
"Geometry and faith," Viola Stegall '27.

(Report by Professor Chester R. Hillard.)

PROBLEMS AND SOLUTIONS.

Edited by B. F. Finkel, Otto Dunkel, and H. L. Olson.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

3113. Proposed by A. S. WIENER, Cornell University.

A man is paying off a mortgage of N dollars, interest i per cent. annually, by monthly installments of a dollars (where $a > iN/1200$, the first month's interest). How many months will it take him to pay off the mortgage?

3114. Proposed by B. R. HEADSTROM, Boston, Mass.

What is the per cent. error in the following supposed method of trisecting an angle? Given the angle ABC with its vertex at B, lay off on its sides the equal lengths BD and BE and draw the line DE. With this latter length as diameter draw a semi-circle on the side opposite to B; also draw arcs of circles with D and E as centers and radii equal to that of the semi-circle cutting the latter in M and N, respectively. Then the lines BM and BN trisect the given angle.

3115. Proposed by H. W. REDDICK, Cooper Union Institute of Technology.

Given a plane triangle and a system of coördinate axes in a 3-space, find the number of ways of placing the triangle so that its vertices lie on the axes.

3116. Proposed by A. A. BENNETT, University of Texas.

Show that if a number is k times the number obtained by reversing the order of its digits, and neither has a zero for leading digit, then $k = 1, 4$, or 9. Show by example that 4 and 9 are actually possible.

3117. Proposed by W. J. SIDIS, New York City.

It is well known that, except for a possible factor 3, all prime factors of $m^2 \pm mn + n^2$ must be of the form $6k + 1$, if m and n are prime to each other. Prove that, if n is an odd prime, all prime factors of

$$x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1},$$

and

$$x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1},$$

excluding a possible factor n, must be of the form $2kn + 1$, provided x and y are prime to each other.

3118. Proposed by HARRY LANGMAN, New York City.

If $n > 2$ and ϵ is a primitive root of $\epsilon^n = 1$, show that

$$\begin{vmatrix} \epsilon & \epsilon^2 & \epsilon^3 & \cdots & \epsilon^{n-1} \\ \epsilon^2 & \epsilon^4 & \epsilon^6 & \cdots & \epsilon^{2(n-1)} \\ \epsilon^3 & \epsilon^6 & \epsilon^9 & \cdots & \epsilon^{3(n-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \epsilon^{n-1} & \epsilon^{2(n-1)} & \epsilon^{3(n-1)} & \cdots & \epsilon^{(n-1)(n-1)} \end{vmatrix} = (-1)^{(n-1)(n-2)/4} n^{(n-2)/2}.$$

3119. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

Through a given point to draw a line meeting the sides BC, CA, AB of a given triangle ABC in the points P, Q, R such that $PQ = kQR$, where k is a given constant.

SOLUTIONS.

2957 [1922, 81]. Proposed by J. L. WALSH, Harvard University.

The envelope of the circles of curvature of a curve is, in part at least, the curve itself. What further curves, if any, are parts of this envelope?

SOLUTION BY THE PROPOSER.

Let us use rectangular coördinates (α, β) for points on the curve which we take as $\beta = f(\alpha)$, (1). The equation giving the circles of curvature is

$$\{x - \alpha + (f' + f'^3)/f''\}^2 + \{y - \beta - (1 - f'^2)/f''\}^2 = (1 + f'^2)^3/f''^2. \qquad (2)$$

We differentiate (2) partially with respect to α taking account of (1), thereby obtaining the equation $f'(x - \alpha) - (y - \beta) = 0$, (3) after we have suppressed the factor $3f'f''^2 - f''' - f'^2f'''$. This factor corresponds to the differential equation satisfied by a circle. If (1) represents a circle, (2) represents the same circle, so no extraneous curves can be introduced.

It remains to eliminate α and β from (1), (2), and (3). Substitution of the value of $(y - \beta)$ from (3) in (2) gives an equation which reduces to $(x - \alpha)^2(1 + f'^2) = 0$. If we are dealing with a real curve, the second factor is never zero, so

$$x = \alpha, \qquad y = \beta, \qquad y = f(x).$$

Hence *the entire envelope of the circles of curvature of a curve is precisely the curve itself.*

2972 [1922, 224]. Proposed by J. H. M. WEDDERBURN, Princeton University.

In *Mathematical Education*, p. 361, B. Branford gives the following approximate construction for the trisection of an angle ABC. Take any point D on AB produced and make $BC = BD$; through C draw a parallel to BA and on it lay off a length $CP = CD$; then ABP is approximately $\frac{1}{3}ABC$. Show that this construction is rendered very much more accurate by laying off on CP a distance $CQ = 2BD$ and taking in place of P the point R which divides PQ in the ratio $5 : 4$. Also show that the error does not exceed $10''$ for angles less than $45°$.

SOLUTION BY OTTO DUNKEL, Washington University.

Let the given angle ABC be denoted by θ and take $BD = BC = 1$. Then $DC = CP = 2\cos\theta/2$. In the first construction the approximation is too large and we shall set angle $ABP = x = \theta/3 + h$, where h is the error. The projection of BCP on BA is $\cos\theta + 2\cos\theta/2$, and hence

$$\tan x = \frac{\sin\theta}{\cos\theta + 2\cos\frac{\theta}{2}}. \tag{1}$$

Also from $h = x - \theta/3$, we have

$$\tan h = \frac{\cos\frac{\theta}{3}\tan x - \sin\frac{\theta}{3}}{\cos\frac{\theta}{3} + \tan x \sin\frac{\theta}{3}}. \tag{2}$$

Inserting in (2) the value of $\tan x$ above and after making certain simple trigonometric reductions we find

$$\begin{aligned}\tan h &= \frac{\sin\frac{2\theta}{3} - 2\sin\frac{\theta}{3}\cos\frac{\theta}{2}}{\cos\frac{2\theta}{3} + 2\cos\frac{\theta}{3}\cos\frac{\theta}{2}}, \\ &= \frac{4\sin\frac{\theta}{12}\sin\frac{\theta}{3}\sin\frac{5\theta}{12}}{4\cos\frac{\theta}{12}\cos\frac{\theta}{3}\cos\frac{5\theta}{12} - 1}.\end{aligned} \tag{3}$$

From the first expression above it is clear that the denominator does not vanish for θ less than or equal to 135° and from either we see that it decreases as θ increases up to this value. Also from the second it is seen that the numerator increases with θ from $\theta = 0$ to $\theta = 216°$. Hence the error h is positive and it increases with θ. It will be found that the denominator is not zero until $\theta = 156°\ 35.72'$, hence h increases from 0 to 90° as θ increases from 0 to 156° 35.72′.

It should be noticed that it is only necessary to consider angles θ less than or equal to 45°. For if θ is greater than 90°, we trisect the supplementary angle θ' and subtract $\theta'/3$ from 60°. If θ is greater than 45°, we trisect the complementary angle θ'' and subtract $\theta''/3$ from 30°.

For $\theta = 45°$ we find from (1) or (3) $h = 28'\ 13.5''$, and the above discussion shows that as θ decreases the error decreases from this amount. If θ is very small and all the angles are in radians, then approximately

$$h = \frac{5\theta^3}{324}. \tag{4}$$

Since $h < \tan h$, it is easy to obtain upper limits for h when the upper limit of θ is given. Thus if $\theta \leqq \pi/4$,

$$h < \frac{\theta^3}{57}. \tag{5}$$

This results from replacing the sine by the angle and computing the rest of formula (3) for $\theta = \pi/4$.

The Second Construction. Here we have $CR = (4CP + 5CQ)/9 = (8\cos\theta/2 + 10)/9$. Setting angle $ABR = x$, we shall here set $h = \theta/3 - x$, since it will appear later that x is too small. Just as before we find

$$\tan x = \frac{9\sin\theta}{9\cos\theta + 8\cos\frac{\theta}{2} + 10}, \qquad \tan h = \frac{\sin\frac{\theta}{3} - \cos\frac{\theta}{3}\tan x}{\cos\frac{\theta}{3} + \sin\frac{\theta}{3}\tan x}. \tag{6}$$

Eliminating $\tan x$ from these two equations and making certain trigonometric transformations, we find

$$\tan h = \frac{16\sin\frac{\theta}{3}\sin^4\frac{\theta}{12}\left(7 + 8\cos\frac{\theta}{6}\right)}{2\cos\frac{\theta}{3}\left(9\cos\frac{\theta}{3} + 4\cos\frac{\theta}{2} + 5\right) - 9}. \tag{7}$$

Here the denominator is positive for $\theta \leqq 180°$ and hence h is positive. In order to examine the right-hand side we separate it into two factors one of which may be written

$$\frac{8 + 7 \sec \frac{\theta}{6}}{2 \cos \frac{\theta}{3} \sec \frac{\theta}{6} \left(9 \cos \frac{\theta}{3} + 4 \cos \frac{\theta}{2} + 5 \right) - 9 \sec \frac{\theta}{6}}. \tag{8}$$

The numerator of (8) increases with θ; the part of the denominator in parentheses decreases as θ increases; $-9 \sec \theta/6$ decreases as θ increases; and finally $\cos \theta/3 \sec \theta/6 = 2 \cos \theta/6 - \sec \theta/6$ also decreases as θ increases. Hence (8) increases as θ increases up to 180°, and, since the other factor of (7), *i.e.*, $16 \sin \theta/3 \sin^4 \theta/12$ increases as θ increases, we see that h increases from 0 as θ increases from 0 to 180°.

The first value of θ for which the denominator is zero is $\theta = 182° 52' 6''$ and for this value $h = 90°$.

When $\theta = 45°$, (6) or (7) gives $h = 9.6''$. If then the angle θ is less than or equal to 45°, the error is less than or equal to this amount. If θ and h are in radians, then for small angles we have approximately

$$h = \frac{10\theta^5}{9 \cdot 6^5}. \tag{9}$$

Just as before by replacing the sine by the angle and computing the remaining parts of (7) for $\theta = 45°$, we find

$$h < \frac{\theta^5}{6403}, \qquad \theta \leqq \frac{\pi}{4}.$$

It is obvious from the above that the second approximation is very much better than the first and that it gives a very high degree of accuracy.

3070 [1924, 206]. Proposed by J. L. RILEY, Tarleton Station, Texas.

A triangle circumscribes the circle $x^2 + y^2 = r^2$ and two of its vertices lie on the circle $(x - a)^2 + y^2 = R^2$; find the condition necessary that the third vertex may also lie on this circle.

SOLUTION BY THEODORE BENNETT, University of Illinois.

Call the first circle C_1 and the second C_2; then C_1 may be written

$$x = r \frac{1 - t^2}{1 + t^2}, \qquad y = r \frac{2t}{1 + t^2}. \tag{1}$$

If we draw tangents to C_1 at the points whose parameters are t and T, it is easily verified that they intersect at

$$x = r \frac{1 - tT}{1 + tT}, \qquad y = r \frac{t + T}{1 + tT}. \tag{2}$$

By substituting these values of x and y in the equation of C_2 and rearranging, we obtain the condition that this point lie on C_2 in the form

$$T^2(\alpha t^2 + r^2) + 2tT(a^2 - R^2) + (r^2 t^2 - \beta) = 0, \tag{3}$$

where $\alpha = (r + a)^2 - R^2$, $\beta = (r - a)^2 - R^2$.

Now consider t fixed; this fixes a tangent to C_1 which cuts C_2 in A_1 and A_2. Through A_1 there is a second line l_1 tangent to C_1, and similarly through A_2 a tangent l_2; the points where l_1 and l_2 touch C_1 are given by (1) for $t = t_1$ and $t = t_2$, where t_1 and t_2 are obtained by solving (3) for T. The intersection of l_1 and l_2 can be written down at once from (2) by substituting the values of $t_1 t_2$ and $t_1 + t_2$, which can be read off from (3); the result is

$$x = r \frac{t^2(\alpha - r^2) - (\beta - r^2)}{t^2(\alpha + r^2) + (\beta + r^2)}, \qquad y = \frac{-2rt(a^2 - R^2)}{t^2(\alpha + r^2) + (\beta + r^2)}. \tag{4}$$

The problem requires the condition that this point lie on C_2.

Substituting these values of x and y in the equation of C_2, we obtain a quartic equation in t, which can be factored in the form

$$(t^2+1)(\alpha t^2+\beta)(a^4+R^4-2a^2R^2-4r^2R^2)=0. \tag{5}$$

The first two factors equated to zero give the four intersections of the two circles (the first giving the circular points at infinity), as may be verified by substituting $t=\sqrt{-\beta/\alpha}$ in (1). At these points the conditions of the problem are obviously satisfied (by improper triangles). To have a triangle none of whose vertices are points of intersection of C_1 and C_2, the third factor of (5) must vanish. Since this puts a condition on C_1 and C_2 but not on t, we see that if there is one such triangle, then there is an infinite number of them, any point of C_2 being a vertex of some one triangle. Setting the third factor of (5) equal to zero and rearranging, we have

$$a^2+r^2=(r\pm R)^2.$$

This shows that if the y-axis cuts C_1 at P, then the distance from P to the center of C_2 must be either the sum or difference of the radii of C_1 and C_2. This gives a necessary and sufficient geometrical condition for the existence of such triangles in any two given circles.

NOTE BY THE EDITORS: The distance a between the centers of the circumscribed and inscribed circles of a triangle $a^2=R^2-2rR$ is derived on page 79 of Chauvenet's *Trigonometry*, 9th ed. It is also shown there that the distance between the centers of the circumscribed and an escribed circle is $a^2=R^2+2r'R$. See also Salmon's *Conic Sections*, page 343, 6th ed. These are therefore necessary conditions. If either one of these conditions is satisfied, it may be easily shown by elementary geometry that there are two isosceles triangles which are inscribed in R and which circumscribe r. Let ABC be one of these triangles and from any point A' of the circle R draw two tangents to the circle r cutting R in B', C', the points A', B', C' being taken in the same order of rotation as A, B, C. It will be shown that $B'C'$ must also be tangent to r. Let AB, AC cut $B'C'$ in M', N', respectively; $A'B'$, $A'C'$ cut BC in M, N, respectively. From A as a center project B, C, B', C' on $B'C'$; from A' as a center project the same points on BC. Then the ranges (M', N', B', C') and (B, C, M, N) are projective, and therefore (N, M, C, B) and (M', N', B', C') are also projective. From this it follows that the three points (NN', MM'), $(MB', N'C)$, $(NC', M'B)$ lie on a straight line. If we call the last two points P and P', then the hexagon $M'P'NMPN'$ has the first five sides tangent to r, and by Brianchon's Theorem the remaining side $N'M'$ (*i.e.*, $B'C'$) must also be tangent. By a slight rearrangement of this proof the two equalities above may be deduced as necessary conditions.

Also solved by MAURICE BAUDIN and J. H. WEAVER.

3074 [1924, 206]. Proposed by J. H. MURPHY, Pittsburgh, Pa.

Two circles of radii R_1, R_2 intersect. A third circle passes through their points of intersection. Find the radius of this third circle, when the sum of the areas of the two crescents cut from it by the other two circles is a maximum.

SOLUTION BY F. HENROTEAU, Dominion Observatory, Ottawa, Canada.

Let C_1, C_2, C_3 be the three circles, A, B, C, D the four points where C_1 and C_2 cut their common diameter and E and F their points of intersection. The third circle C_3 may be defined by E, F and a third point K anywhere on AD.

Suppose that K moves on AD from $-\infty$ to $+\infty$. We see easily that when K moves from $-\infty$ to A the sum of the areas of the two crescents is increasing.

When K is between A and D, the sum of the areas of the two crescents is equal to the area of C_3 plus the area of the crescent common to C_1 and C_2 which is a constant. From A to the intersection of AD with EF the area of C_3 decreases, from this last point to D it increases again—so it is the same for the sum of the areas of the two crescents. When K moves from D to $+\infty$, the sum of the areas is decreasing. There are thus two maxima and a minimum when $R=R_1$, $R=R_2$ and $R=EF/2$ respectively, for the sum of the areas of the two crescents.

Also solved by C. K. ROBBINS.

3079 [1924, 254]. Proposed by J. J. QUINN, Pittsburgh, Pa.

The line OP is equal to and coincident with the straight line segment AB. As O describes AB uniformly, OP rotates uniformly through 180°. Determine (*a*) the locus of P; (*b*) the length of the curve for a complete revolution; (*c*) the area of a loop.

Solution by P. J. da Cunha, Lisbon University.

Take as the x-axis the line AB and for the y-axis the perpendicular to it at A. Let $AB = a$ and set angle $BOP = t$. From the definition of the locus in the problem we have $OA/a = t/\pi$, and hence it results that

$$x = \frac{at}{\pi} + a\cos t, \qquad y = a\sin t,$$

are the parametric equations of that locus. By a change of axes it will be seen that the curve is the epitrochoid inverted. The location of the maximum breadth of the loop on AB is found by setting

$$\frac{dx}{dt} = a\left(\frac{1}{\pi} - \sin t\right) = 0,$$

which gives $t = 18° 33' 39''$ or $161° 26' 21''$. The loop is symmetric with respect to the perpendicular bisecting AB.

(*b*) For the length of the part of the loop above AB we have

$$s = \frac{2a\sqrt{1+\pi^2}}{\pi}\int_0^{\pi/2}\sqrt{1 - k\sin t}\,dt, \qquad k = \frac{2\pi}{1+\pi^2}.$$

Developing $\sqrt{1 - k\sin t}$ by the binomial theorem and integrating, we obtain

$$s = \frac{a\sqrt{1+\pi^2}}{\pi}\Big[\pi - k - \left(\frac{1}{2}\right)^2\frac{\pi k^2}{4} - \frac{2k^3}{4\cdot 6} - \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{5\pi k^4}{6\cdot 8} - \cdots$$
$$- \left(\frac{1\cdot 3\cdots(2m-1)}{2\cdot 4\cdots 2m}\right)^2\frac{(2m+1)(2m+3)\cdots(4m-3)}{(2m+2)(2m+4)\cdots 4m}\pi k^{2m}$$
$$- 2\frac{(2m+3)(2m+5)\cdots(4m-1)}{(2m+2)(2m+4)\cdots(4m+2)}k^{2m+1} - \cdots\Big].$$

A calculation of the first ten terms gives the approximate value

$$s = 2.474a.$$

(*c*) The area of the part of the loop above AB is given by

$$A = 2a^2\int_0^{\pi/2}\left(\frac{t}{\pi} - \frac{1}{2} + \cos t\right)\cos t\,dt,$$
$$= .934a^2 \text{ (approximately).}$$

To find the area of the entire loop the double point below AB must be found. This is given by $t = -42° 34.03'$. The area is then found to be $1.3467a^2$.

3080 [1924, 255]. Proposed by C. N. Schmall, New York City.

Given a cyclic quadrilateral $ABCD$ such that its diagonals AC, BD intersect at right angles in P. If O be the center of the circle, and perpendiculars be dropped on the sides of the quadrilateral from O and P, show that the eight feet of these perpendiculars are concyclic.

Note: The circle on which these points lie is somewhat analogous to the *nine-point* circle of a triangle. Its center is the middle point of OP.

Solution by C. H. Chepmell, Hove, Sussex, England.

Bisect OP in M. Let OS and PT be the perpendiculars on AB; let R be the radius of the circle $ABCD$. Then $MS = MT$. Draw OD and SP. Then, because APD is a right angle and S is the mid-point of AD, $SP = SD$; and $OS^2 + SP^2 = OS^2 + SD^2 = R^2$. Hence $MS = MT = \sqrt{(R^2/2) - OM^2} = L$ (a constant). Thus all the eight points such as S and T lie on the circle with center at M and radius L.

Also solved by Theodore Bennett, E. P. Bugdenoff and Alex Wiesner.

NOTES AND NEWS.

Readers are invited to contribute to the general interest of this department by sending items to R. W. BURGESS, c/o Western Electric Co., 195 Broadway, New York City.

The one hundredth anniversary of the announcement by Sadi Carnot of the second law of thermodynamics was celebrated in New York City under the auspices of the Engineering Foundation, on December 4, 1924. Addresses were delivered by Professor M. I. PUPIN, of Columbia University, and Dr. W. L. EMMET, of the General Electric Company, on Carnot's principle.

On the occasion of the centenary of the Franklin Institute, the University of Pennsylvania conferred honorary degrees on Sir W. H. BRAGG, Mr. W. C. L. ENGLIN, Dr. CHARLES FABRY, Sir CHARLES PARSONS, Dr. E. W. RICE, Jr., and Dr. PIETER ZEEMAN.

The General Electric Company has appropriated to Union College a fund of $25,000 in memory of Dr. C. P. STEINMETZ. The income is to be used for scholarships.

The Rensselaer Polytechnic Institute of Troy, the oldest school of science and engineering in any English-speaking country, celebrated on October 3–4, 1924, the centenary of its foundation. On this occasion the Institute conferred the honorary degree of Doctor of Science on Professor A. A. MICHELSON, President of the National Academy of Sciences.

Professor H. A. WILSON, of Rice Institute, has been appointed professor of natural philosophy at Glasgow University.

Dr. G. N. BAUER has been appointed associate professor of mathematics at the University of New Hampshire.

Professor H. B. MITCHELL, of Columbia University, has resigned to enter business.

Instructor VICTOR DOUSHKESS, of Lafayette College, has been promoted to an assistant professorship of mathematics.

Associate Professor O. J. RAMLER, of Catholic University, has been promoted to a full professorship.

Mrs. R. B. MONTGOMERY, of Lynchburg College, has been promoted to an assistant professorship of mathematics.

Mr. L. W. JARMON has been appointed head of the department of mathematics at Chicora College, Columbia, S. C.

At the University of South Carolina, Mr. W. L. WILLIAMS has been appointed adjunct professor and Mr. R. L. JONES instructor.

Professor E. A. BAILEY, of LaGrange College (Georgia), has been appointed dean.

The Founders' Day address by Professor R. M. WINGER as retiring president of the Chapter of Phi Beta Kappa at the University of Washington was printed in full in the *Phi Beta Kappa Key* for October, 1924.

THE MAY MEETING OF THE IOWA SECTION.

The thirteenth regular meeting of the Iowa section of the Mathematical Association of America was held in Central Building, Iowa State College, Ames, Iowa, on May 2 and 3, 1924. There were forty-three present, including the following twenty-four members of the Association: Julia T. Colpitts, Marian E. Daniells, R. M. Deming, C. W. Emmons, Iva Ernsberger, Fay Farnum, C. Gouwens, F. M. McGaw, J. V. McKelvey, Martha McD. McKelvey, I. F. Neff, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, W. J. Rusk, J. R. Sage, E. R. Smith, G. W. Snedecor, A. Helen Tappan, J. S. Turner, F. M. Weida, C. W. Wester, R. Woods.

On Friday evening the members and their friends enjoyed dinner together, arrangements being in charge of the mathematical fraternity Pi Mu Epsilon; Professor E. R. Smith acted as toastmaster.

Chairman McGaw presided at both the Friday afternoon and Saturday morning sessions, being relieved by vice-chairman McKelvey for a time.

Officers elected for this year are: Chairman, Professor E. R. SMITH, Iowa State College; Vice-chairman, Professor I. F. NEFF, Drake University; Secretary-treasurer, Professor J. F. REILLY, University of Iowa.

The next meeting of the Section will be held at Iowa State Teachers' College, Cedar Falls, Iowa, on May 1 and 2, 1925.

The following sixteen papers were presented:

(1) "Reduction of the equation of the general conic section" by Professor W. J. RUSK.

(2) "The preparation of high school teachers of mathematics in Iowa" by Professors E. W. CHITTENDEN and F. M. WEIDA.

(3) "On the meaning, evaluation and application of the refund integral" by Professor F. M. WEIDA.

(4) "The zeros of the function $f_k(z) \equiv \sum_{n=0}^{\infty} n^k \frac{z^n}{n!}$, where k is a positive integer" by Professor JULIA T. COLPITTS.

(5) "Note on the average number of brothers and of sisters of the boys in families of n children" by Professor H. L. RIETZ.

(6) "Relation of surface-level of a rotating fluid to the shape of the container" by Professor E. S. ALLEN.

(7) "Determination of the foci and directrices of a conic section" by Professor ALLEN.

(8) "Construction of tangents of an ellipse" by Professor ALLEN.

(9) "A problem in mathematical teaching" by Professor F. M. MCGAW.

(10) "The isobaric differential equation in an elementary course" by Professor J. F. REILLY.

(11) "A new interpolation formula" by Professor REILLY.

(12) "A theorem in spherical trigonometry" by Professor J. S. TURNER.